Examples for the failure of a Heine-Borel type theorem in metric spaces other that $\mathbb{R}^n$:

In $\mathbb{Q}$, there is a bounded sequence that has no convergent subsequence: For instance the sequence $x_1 = 1$, $x_2 = \frac{3}{2}$, $x_3 = \frac{17}{12}$, ..., which is recursively defined by $x_{n+1} := \frac{1}{2}(x_n + 2/x_n)$ defines a bounded sequence in $\mathbb{Q}$, which, considered as a sequence in $\mathbb{R}$, has the limit $\sqrt{2}$ (as can be proved with a little labor, but is not the main issue here). Considered as a sequence in $\mathbb{Q}$, it does not have a limit in $\mathbb{Q}$, nor does any subsequence.

In a discrete metric space X, *all sequences are bounded*. Now if we take a discrete metric space with infinitely many elements, we can construct a sequence that has no repeat elements (i.e., a function $\mathbb{N} \to X$ that is 1-1). Such a sequence has no cluster point.

The following examples are more interesting:

Hwk 3.9X1: Let $X = C^0[-1,1]$ with the max distance. Define $f_n \in X$ by $f_n(x) := nx/\sqrt{1+(nx)^2}$. The sequence (f_n) is bounded, since it lies in $B(\text{zerofunction}, 1)$. But prove that it does not have a convergent subsequence. *Hint: Assume some subsequence does have a limit f. Use the evaluation functions $\text{ev}_x : C^0[0,1] \to \mathbb{R}$ defined by $\text{ev}_x(f) := f(x)$ to find out what f would have to be and obtain a contradiction.*

Hwk 3.9X2: Let $X = C^0[-1,1]$ with the max distance. Define $f_n \in X$ by $f_n(x) := nx/(1+(nx)^2)$. The sequence (f_n) is bounded, since it lies in $B(\text{zerofunction}, 1)$. But prove that it does not have a convergent subsequence. *Hint: Same as before; but note that the contradiction to be obtained is of a different kind this time. You may want to calculate $d(f_n, 0)$.*

Hwk 3.9X3: Let $X = C^0[-1,1]$ with the max distance. Define $f_n \in X$ by $f_n(x) := \sin nx$. The sequence (f_n) is bounded again. But prove that it does not have a convergent subsequence. *Hint: Assume there is a convergent subsequence with limit f. Use uniform continuity of f to prove:*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall g \in B(f,\varepsilon)\ \sup_{[-\delta,\delta]} g - \inf_{[-\delta,\delta]} g < 3\varepsilon \qquad (EC)$$

Note that δ must not depend on g (but may depend on f). Derive a contradiction by having $f_n \notin B(f,\varepsilon)$.

Connectedness:

Example and comment: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$, because the complement of a singleton in $\mathbb{R}$ is not connected whereas the complement of a singleton in $\mathbb{R}^n$ is connected, as can be easily verified. — It is true, but much more sophisticated to prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^n$, except if $n = m$. The tools to prove this theorem belong in the area of algrebraic topology, and they are beyond 447/448. — Another famous theorem in this area is the Jordan curve theorem: If C is a simple closed curve in $\mathbb{R}^2$, then its complement is not connected, but is the union of exactly two connected subsets of $\mathbb{R}^2$. Here, 'simple closed curve' means the image of a continuous and injective function from $\{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ to $\mathbb{R}^2$. This theorem, intuitively plausible, is also quite sophisticated. Regardless of whether you ever study a proof of these results in a class or not, consider the contents of this paragraph as core part of the GenEd of a mathematician.

Hwk 3.10.X1: Show that the only connected subsets of a Cantor set are singletons.

Hwk 3.10.X2: Is $\mathbb{R}^2 \setminus \mathbb{Q}^2$ connected or not? Explain.

Metric Completeness:

The following theorem is the simplest among a collection of power tools of advanced calculus. Its applications include the Newton method for solving systems of equations, the local existence and uniqueness theorem for ordinary differential equations, and similar results for a variety of partial differential equations.

Theorem: (Banach's fixed point theorem, aka contraction mapping principle) *Suppose $X \neq \emptyset$ is a complete metric space and $f : X \to X$ is a contraction, i.e., it satisfies $d(f(x), f(y)) \leq \vartheta d(x, y)$ for some constant $\vartheta < 1$. Then, in X, there exists exactly one solution x to the equation $f(x) = x$.*

Proof idea: Choose any x_0 in X. Define recursively $x_n := f(x_{n-1})$. Show that (x_n) is a Cauchy sequence, by getting a recursive estimate for $d(x_k, x_{k-1})$ and using the triangle inequality. Its limit solves the equation. Finally prove uniqueness.

Experiment: As an application for $X = [0, 1]$, you can solve the equation $\cos x = x$ by repeatedly hitting the cos key on your pocket calculator.

Tourism Corollary for X = Knoxville**:** If you lay down a city map of Knoxville on the ground anywhere within Knoxville, then there is exactly one point in Knoxville that lies right beneath its corresponding representation on the map. This is true even if the map is (partially) folded up, lying face down or is of the special patented kind, which introduces a distortion to the effect of having a larger scale for the center than for the suburbs.

Hwk 3.11X1: Work out the proof details based on the above idea.

Hwk 3.11X2: Give an example for $X = \mathbb{R}$ to the effect that BFPT becomes false if the hypothesis '$d(f(x), f(y)) \leq \vartheta d(x, y)$ for some constant $\vartheta < 1$' is replaced by the weaker hypothesis $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$.

Hwk 3.11X3: Show that the (false) BFPT from the previous exercise becomes true again, if we throw in the extra hypothesis that X is compact. Namely show: If X is a compact metric space, and $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$, then there exists exactly one solution x to the equation $x = f(x)$. *Hint: Consider the sets $X_0 := X$, $X_n := f(X_{n-1})$ and study the intersection $K := \bigcap X_n$. In particular show $f(K) = K$.*

Comment: The last two exercises serve the purpose of better understanding the first; in stark contrast to BFPT, they seem to have little application themselves.

Theorem: *$C^0[a, b]$, together with the max distance is a* complete *metric space.*

This is easy to prove (and for that matter generalizes to $C^0(K)$ where K is any compact subset of $\mathbb{R}^n$, and with minor modifications to yet vaster generality):

Proof: Let (f_n) be a Cauchy sequence in $C^0[a, b]$:

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall m, n \geq n_0 : \max_{x \in [a,b]} |f_n(x) - f_m(x)| = d(f_n, f_m) < \varepsilon$$

It follows that for each $x \in [a, b]$, the sequence $(f_n(x))_n$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, this Cauchy sequence has a limit, and we call it $f(x)$. In other words, we have defined a function $f : [a, b] \to \mathbb{R}$ that assigns to each $x \in [a, b]$ the limit $\lim_{n\to\infty} f_n(x)$. We next aim to prove that this function is indeed continuous and hence lies in $C^0[a, b]$. After that we will show that $d(f_n, f) \to 0$, i.e., that f is indeed the limit of the Cauchy sequence (f_n).

First the proof that f is continuous at each x: Let $\varepsilon > 0$ and choose n_0 so large that $d(f_n, f_m) < \frac{\varepsilon}{3}$ for all $m, n \geq n_0$. Then $|f_n(t) - f_m(t)| < \frac{\varepsilon}{3}$ for all t. Now estimate

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f_n(y)| < \frac{2}{3}\varepsilon + |f_m(x) - f_m(y)|$$

We may let $n \to \infty$ in this estimate, since we know already that the sequence of real numbers $f_n(t)$ converges for every t; and we get

$$|f(x) - f(y)| \leq \frac{2}{3}\varepsilon + |f_m(x) - f_m(y)|$$

This is true for every fixed $m \geq n_0(\varepsilon)$, for instance we can just take $m = n_0$. Since this f_m is continuous, we can find a $\delta > 0$ such that $|x - y| < \delta$ implies $|f_m(x) - f_m(y)| < \frac{\varepsilon}{3}$. With this choice of δ, we then get immediatley that $|f(x) - f(y)| < \varepsilon$, rovided $|x - y| < \delta$. We have showed the contnuity of f.

Next we want to show $d(f_n, f) \to 0$. We rewrite the Cauchy sequence property:

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall m, n \geq n_0 \, \forall x \in [a, b] : |f_n(x) - f_m(x)| < 0.9\varepsilon$$

Taking the limit $n \to \infty$ in this estimate, we get

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall m \geq n_0 \, \forall x \in [a, b] : |f(x) - f_m(x)| \leq 0.9\varepsilon < \varepsilon$$

But this means $d(f, f_m) \leq 0.9\varepsilon < \varepsilon$, and the theorem is proved.

Example: $C^0[a, b]$ can also be equipped with the metric $d_1(f, g) := \int_a^b |f(x) - g(x)| \, dx$. With this metric, the space is NOT complete.

Hwk 3.11X4: Show that the sequences in homeworks 3.9.X1 and 3.9.X2 are Cauchy sequences with respect to the metric d_1. *Hint:* The second sequence is easier, b/c you can just estimate $\int |f - g| \leq \int |f| + \int |g|$, and it will be good enough whereas in the first example you should use more care in estimating the difference. Obviously you are allowed to use elementary calculus knowledge here to handle the integrals.

Compact sets in $C^0[a, b]$

Have another look at condition (EC) in the hint for problem 3.9.X3. The punchline was that the modulus of continuity (i.e., the δ chosen as a function of ε such that the definition of continuity is satisfied for this choice of δ) was the same for all functions g in the ball.

Definition: *A set $S \subset C^0[a, b]$ is called equicontinuous iff*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall g \in S : |x - y| < \delta \Longrightarrow |g(x) - g(y)| < \varepsilon$$

This definition includes uniform continuity, which is anyways automatically satisfied, because $[a, b]$ is compact. But the crucial issue is that δ does *not* depend on $g \in S$ (because the $\forall g$ quantifier follows the $\exists \delta$ quantifier, rather than preceding it).

Here is a famous theorem that characterizes compact sets in $C^0[a, b]$ in a similar way as Heine Borel characterizes compact sets in $\mathbb{R}^n$:

Theorem: (Arzelà-Ascoli) *A set $S \subset C^0[a, b]$ is compact if and only if it is closed, bounded, and equicontinuous.*

We'll see later if we get around proving this theorem. I think it is desirable to include it in the course, but it is not required core material. Key proof ingredients are Heine-Borel for the sets $\mathrm{ev}_x(S) \subset \mathbb{R}$, selection of a countable dense subset of $[a, b]$ from which the x's will be chosen, and a tricky way of successive selection of subsequences known under the label 'diagonal sequence argument'. The Arzelà-Ascoli theorem is another core power tool of modern analysis, and it spawns a whole family of corollaries for other metric spaces consisting of functions.