Homework
UTK – M531 – Ordinary Differential Equations I
Fall 2004, Jochen Denzler, TR 11:10–12:25, Ayres 309B

The first 8 problems serve multiple purposes: Familiarize yourself with key examples, in particular from a modelling point of view, get a sneak preview of later material, appreciate the need to get qualitative *information about solutions of an ODE, with or without the ability to come up with formula solutions, and along the way review some techniques learned in the sophomore ODE course.*

1. Exponential Growth Give physical / biological interpretations to the variables y, t, and the parameter k (positive or negative) in the ODE

$$y'(t) = ky(t)$$

if the ODE describes (a) radioactive decay, (b) Newton's law of cooling, (c) population growth. Write down the set of all solutions to this ODE (called general solution in the sophomore level courses). In case (a), express the half-life of the radioactive substance in terms of the/a variable(s)/parameter present in the ODE.

This is kind of a review of sophomore ODE material. See any textbook or my web notes for this level, or me, with any difficulties here.

The flow function $(t, y) \mapsto \phi(t, y)$ is defined as follows: $\phi(t, y_0)$ is the value $y(t)$ of the (unique) solution of the ODE subject to the initial condition $y(0) = y_0$. Give a formula for the function ϕ.

2. Two Step Radioactive Decay A nucleus A decays, with a decay constant k_1 into a nucleus B. This nucleus is itself radioactive and decays, with a decay constant k_2, into a stable nucleus C. Set up the differential equations for the concentrations $A(t)$, $B(t)$, $C(t)$ of the respective nuclei as a function of time, find the solution, expressed in terms of the initial concentrations A_0, B_0, C_0. Determine the large-time asymptotic behavior of the ratio $A(t)/B(t)$, distinguishing the cases $k_2 > k_1$ and $k_2 < k_1$. (Never mind the physically irrelevant case where $k_1 = k_2$.)

Note: asymptotic behavior refers to the limit as $t \to \infty$, provided this limit exists and is neither 0 nor infinity. Otherwise, find a simple function (in our case ae^{bt}) such that the limit (as $t \to \infty$) of $(A(t)/B(t))/(ae^{bt})$ is 1.

3. Predator Prey Model (Lotka–Volterra Equations) Find all equilibrium solutions (stationary solutions) for the confined predator-prey model

$$\begin{aligned} R' &= aR - bRF - eR^2 \\ F' &= -cF + dRF - fF^2 \end{aligned}$$

Choosing $e = f = 0$, $a = b = c = d = 0.3$, sketch the 'phase portrait' (aka vector field) of the predator prey model. Also sketch one or two solution curves (aka integral curves of the vector field).

4. Modelling a Leaky Bucket Derive the leaky bucket ODE

$$y' = -k\sqrt{y}$$

from energy considerations, neglecting motion inside the bucket (appropriate for a small leak) and viscosity: The potential energy of a thin 'slab' of water at the top that leaves the bucket, gets converted in kinetic energy of the water that flows out. Make sure to express k in terms of accessible (physical and geometric) quantities. Solve the ODE. How long, depending on the initial height, does it take until the bucket is empty?

5. Mathematical Pendulum The mathematical pendulum obeys the equation

$$\ddot{x} + \tfrac{g}{\ell}\sin x = 0$$

where x denotes the angle as measured from the vertical-down position.

Show that for a particular choice of $a \neq 0$ (which?), and arbitrary t_0, a solution is given by $x = 4\arctan\exp(a(t-t_0)) - \pi$. Sketch a graph of this function and describe the particular motion of the pendulum that is given by these solutions. *Note that* $\sin 4\arctan u$ *is a rational function of* u*, and you can find out which, if you use the double angle formulas* $\sin 2\varphi = 2\sin\varphi\cos\varphi$, $\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$ *twice.*

Unless you are familiar with the so-called elliptic functions usually denoted by sn, cn, dn (and I certainly don't expect such familiarity) you will not be able to write down formulas for a general solution to the mathematical pendulum equation.

Multiply the equation by $\dot{x}$ and integrate, such as to obtain a 1st order ODE instead of the second order ODE. What name does the physicist call the constant of integration thus obtained? (You may prefer also to multiply with the constant $m\ell^2$, where m is the mass of the pendulum, to obtain a more clearly visible meaning of the constant of integration.)

6. Mathematical Pendulum II Write the 2nd order ODE of the previous problem as a 1st order system, sketch a phase portrait, making sure to enter the equilibria and the special solution explicitly calculated there into the figure.

How can the 1st order ODE that you obtained in the previous problem by integrating be represented in the phase portrait?

7. Genesis of the Bessel Equation In the study of oscillations of a circular membrane, the partial differential equation $\Delta w = -\omega^2 w$ arises, where the Laplace operator is $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. *Using the chain rule to transform it in polar coordinates, one obtains* $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$. If you look for solutions w of the special form $w = u(r)\cos n\varphi$, which ODE must u satisfy?

8. Bessel Zeros: Qualitative Heuristics For sufficiently large r (say $r > n$), compare the Bessel equation with the equation of the damped harmonic oscillator; argue based on physical intuition pretending that r should be called 'time', and give a mechanical interpretation. Explain heuristically why the solutions should be oscillatory for large r. Based on heuristic approximations, what approximate distance between adjacent zeros of solutions would you expect?

9. Bessel Zeros: A Rigorous Proof *The heuristics of the previous problem can be made rigorous by means of the following calculations, which, when seen first, will seem miraculous and unmotivated. Just trust me and follow the roadmap. We take the Bessel eqn with* $n = 0$*:*

The substitution $U = r^{1/2}u$ serves the purpose to get an ODE for U that does not contain a first derivative. Find the ODE for U. The ODE $v'' + v = 0$, whose solutions are known, is introduced as an auxiliary equation for comparison with the eqn for U.

Note that any solution $v = a\cos r + b\sin r$ of $v'' + v = 0$ can also be written in the form $v = c\sin(r - r_0)$.

Assume (for the purpose of deriving a contradiction) that some solution u is positive on a certain interval $[r_0, r_0 + \pi]$ of length π and consider the function $U'v - Uv'$, with the comparison function $v = \sin(r - r_0)$. Calculate $\int_{r_0}^{r_0+\pi}(U'v - Uv')'\,dr$ in two ways: (a) with the fundamental theorem of calculus, (b) using the ODEs for U, v.

The contradiction arises when one way of calculation yields a positive quantity, whereas the other way yields a negative quantity.

Having shown that U and hence u cannot be positive on any interval of length π, you conclude that u must have infinitely many zeros, with distance $\leq \pi$ apart.

Note: In the sophomore ODE course, you have encountered the Wronskian $U'v - Uv' = \det\begin{bmatrix} v & U \\ v' & U' \end{bmatrix}$ *of solutions* v, U *of one and the same 2nd order linear ODE. Here we have used a 'mixed' Wronskian of solutions* v, U *of different, but similar ODEs as the key piece of magic in our proof.*

Note: The method of proof, namely to integrate the derivative of a Wronskian to get sign information, is called a Sturm comparison argument. We'll return to it later. The only reason to have it as a preview already now is that I don't want to leave you with only vague heuristis from #8.

10. **Trying Out Some Proof Ideas** Consider the IVP $\dot{x} = x$, $x(0) = 1$. Write down the solution explicitly.

Next write this IVP as an integral equation as outlined in the proof sketch for the Picard-Lindelöf existence and uniqueness theorem. Give formulas for the approximate solutions $x^{[0]}$, $x^{[1]}$, $x^{[2]}$, $x^{[3]}$, etc, obtained in the Picard-Lindelöf iteration. Graph them, together with the exact solution, over the interval $[-1.5, 1.5] \ni t$, in one figure.

Likewise, calculate the piecewise linear approximate solutions $x^{[h]}$ obtained from the explicit Euler method for $h = 1$, $h = 0.5$, and $h = 0.25$ and plot these functions in one figure together with the exact solution, over the interval $[-1, 1] \ni t$. (I've taken license in using the symbol $x^{[?]}$ in two different meanings.)

11. Find a continuous function f that satifies $f(x) = \int_0^x (1 + f(t)^2)\,dt$ for sufficiently small x.

12. **Reconstructibility of the Past: Theory and Modelling** If you find an empty bucket with a leak, can you determine when it was full? (Is the difficulty a theoretical or a practical one?) Which theorem is illustrated by this problem? — An isotope A decays into a stable isotope B with a half-life of 10 hours (Let's say by β decay, which carries away only a negligible mass). You are told that many weeks ago somebody prepared 100g of 90% pure A. Now you find a residue of (almost) 100g of isotope B, with no trace of A left. Can you find out when the 100g of isotope A was prepared? How does this question differ from the leaky bucket question? — Can you also play the devil's advocate and challenge the validity of the distinction just found?

13. **Good Modelling Consistence Properties for Lotka–Volterra** For the predator prey model

$$\begin{aligned} \dot{R} &= aR - bRF - eR^2 \\ \dot{F} &= -cF + dRF - fF^2 \end{aligned}$$

with $a, b, c, d > 0$, $e, f \geq 0$, show that (a) $R(t) = 0$ for all t, provided $R(0) = 0$; (b) $F(t) = 0$ for all t, provided $F(0) = 0$; (c) $R(t) > 0$ and $F(t) > 0$ for all t, provided $R(0) > 0$ and $F(0) > 0$. (Here, 'for all t' is to be understood as 'for all t in the maximal interval of existence'.)

The title is 'modelling consistence properties', because, if the mathematics allowed for the ODE to have solutions starting with a positive number of foxes and rabbits but then ending up with a negative number after some time, the conclusion would have to be that these ODEs are a really poor (fishy or lousy, but certainly not foxy) model.

14. An Integral of Motion Based on part (c) of the previous problem, we here consider as the phase space of the predator prey model only the positive quadrant $R > 0$, $F > 0$. In the predator prey model, with $e = f = 0$, a stroke of genius has invented the function $V(R, F) := dR + bF - c \ln R - a \ln F$, defined for $R, F > 0$. Show that miraculously the function $t \mapsto V(R(t), F(t))$ for solutions $(R(\cdot), F(\cdot))$ to the ODE, is a constant function.

I'm not aware of an official name for this function V, but your calculation shows an analogy to, say, the total energy in certain mechanical systems, which is a function E of the phase space variables (positions and velocities of the particles), and for the actual movements of the particles (i.e., with the actual solutions of the ODE being plugged into this function), it becomes a constant in time. So just to stress this analogy in your mind, I invent the name 'bionergy' for V, for the sole usage in this problem.

Sketch the 'isobionergics' (lines of constant bionergy) in phase space. What information do they give about solutions of the ODE?

Using V in a similar way as $1 + \|x\|^2$ in Cor. 1.2.5, prove that solutions to the predator prey model exist for all positive times, even when $e, f \geq 0$ don't necessarily vanish.

15. Phase Portraits; When are two the Same? The harmonic oscillator $\ddot{x} + \omega^2 x = 0$ can be written as a first order system in the following two ways (and in many other ways as well):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x} &= \omega y \\ \dot{y} &= -\omega x \end{aligned}$$

How (if at all) do the phase portraits differ? — Suggest a definition when two phase portraits should be considered as 'the same'. —*(Chicone Ex 1.8)*

16. Calculating a Flow Calculate the flow map $(t, x, y) \mapsto \phi(t, x, y)$ of the system

$$\begin{aligned} \dot{x} &= y^2 - x^2 \\ \dot{y} &= -2xy \end{aligned}$$

Note: A problem like this may be rather intractable already. What makes this one much easier is that the formulas become simpler in terms of the complex coordinate $z = x + iy$ instead of the real coordinate (x, y). —*(Chicone Ex 1.8)*

17. A Dimensional Effect Prove: An equation $\dot{x} = f(x)$ with $x \in \mathbb{R}$ cannot have non-constant periodic solutions.

This a simple example of the principle that low dimension of the phase space limits the complexity of the solutions. The more interesting case to study later will be what limitations on the behavior of solutions a 2-dimensional phase space can set.

18. Picard–Lindelöf Quantitatively: Redo the proof of the existence/uniqueness theorem for the specific case

$$x(t) = \int_0^t (1 + x(\tau)^2)\, d\tau$$

in the complete metric space (ball in the Banach space) $C^0([-T,T] \to \bar{B}_r(0))$, optimizing T and r such as to get the largest possible interval of existence out of the proof. Compare with the true maximal interval of existence. (The ODE can be solved explicitly.)

The space

$$\{f : [-T,T] \to \mathbb{R} \mid f(t) = t\, g(t)\,;\ g \in C^0([-T,T] \to \mathbb{R})\}$$

with the norm $\|f\|_1 := \sup_{t\in[-T,T]} |f(t)/t|$ is also a Banach space. (Why?) — Redo the contraction mapping argument in a ball of radius r in this Banach space, again optimizing T and r such as to get the largest possible interval of existence out of this proof. (Remember that the contraction constant can be any $\vartheta < 1$ and needn't be the $\frac{1}{2}$ I chose in 1.3.1 of the lecture.

In either case, sketch the set in the (t,x) plane in which the graphs of those functions can lie that are in the ball of the respective Banach space.

I do not mean to allege that this is the best method to get good *estimates for the maximal interval of existence; rather I suggest it as a good way to study the Picard-Lindelöf proof and the Banach space formalism in detail, and to work with different norms, as the task may suggest as convenient.*

19. Insufficient 'Contraction' for Banach's Fixed Point Theorem: $\mathbb{R}$ is a complete metric space. Find a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $|f(x) - f(y)| < |x - y|$ unless $x = y$, but which has no fixed point.

Show on the other hand that the 'weak contraction property' $|f(x) - f(y)| < |x - y|$ unless $x = y$ is sufficient to salvage Banach's fixed point theorem in a *compact* metric space X. *Hint for this one: Consider $Z := \bigcap_n f^n(X)$ with $f^n := f \circ f \circ \cdots \circ f$ (n times); show that Z contains exactly one point.*

20. A Weighted Norm, for Global Estimates: If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies a Lipschitz condition on all of $[-T,T] \times \mathbb{R}^n$

$$|f(t,x) - f(t,y)| \le L|x-y| \quad \text{for all} \quad x, y \in \mathbb{R}^n\,,\ t \in [-T,T]$$

then the solution to $\dot{x} = f(t,x)$, $x(0) = x_0$ exists for all $t \in [-T,T]$ by (a variant of) the extensibility theorem. However, fixed point iteration in $C^0([-T_0,T_0] \to \mathbb{R}^n)$ with the usual norm still requires a smallness condition on T_0 to be a contraction. (Namely, how small must T_0 be?)

Take a different norm in $C^0([-T,T] \to \mathbb{R}^n)$, namely $\|x(\cdot)\|_a := \max\{|x(t)| \exp(-a|t|) : t \in [-T,T]\}$. This norm makes $C^0([-T,T] \to \mathbb{R}^n)$ into a Banach space, too. Show that for appropriate a (which?), the fixed point iteration is a contraction, with the whole time interval $[-T,T]$ allowed.

21. Mathematical Pendulum with Damping How does the phase portrait of the mathematical pendulum $\ddot{x} + \sin x = 0$ change if a small friction is introduced: $\ddot{x} + \delta\dot{x} + \sin x = 0$? You may use heuristic answers, physical intuition, experimental mathematics or rigorous arguments, provided you declare them correctly as what they are.

22. **Coincidential Uniqueness** The vector field f on $\mathbb{R}$ given by $f(0) = 0$, $f(x) = |x|^{1/2}\sin(1/x)$ for $x \neq 0$, is continuous, but is not Lipschitz in any neighbourhood of $x = 0$. Show that nevertheless the initial value problem $\dot{x} = f(x)$, $x(0) = 0$, has only the trivial solution $x(t) \equiv 0$.

23. **Planar 1-body Kepler Problem** Consider the ODE

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -\boldsymbol{x}/|\boldsymbol{x}|^3 \end{aligned} \qquad \text{where} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} u \\ v \end{bmatrix}$$

on $U := \{(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^4 \mid \boldsymbol{x} \neq \boldsymbol{0}\} \subset \mathbb{R}^4$. It describes the motion of a planet around the sun, neglecting the reaction force of the planet on the sun, hence considering the sun as fixed at $\boldsymbol{x} = \boldsymbol{0}$.

Do there exist equilibrium solutions?

Show that the following two quantities are constants of motion: $L := xv - yu$ and $E := \frac{1}{2}|\boldsymbol{v}|^2 - |\boldsymbol{x}|^{-1}$. L denotes the angular momentum (per unit mass) of the planet, and E denotes its total energy (per unit mass).

Show (kinematically) that $E \geq -1/|\boldsymbol{x}| + L^2/(2|\boldsymbol{x}|^2)$. — *When I say 'kinematically', I mean: by using only functions defined on U without reference to the ODE (the vector field on U).*

Show that the joint-level set $\{(\boldsymbol{x}, \boldsymbol{v}) \in U \mid L(\boldsymbol{x}, \boldsymbol{v}) = L_0\,,\ E(\boldsymbol{x}, \boldsymbol{v}) = E_0\}$ is compact, for any choice $E_0 < 0$, $L_0 \in \mathbb{R} \setminus \{0\}$.

What does this imply for the existence time of solutions? Can you salvage your conclusion even if $E_0 \geq 0$ (still assuming $L_0 \neq 0$)? — In very practical geometric terms, which type of initial conditions are excluded by the assumption $L_0 \neq 0$?

Using heuristic arguments, which dimension should the above-mentioned joint-level sets have? — Which advanced calculus theorem makes the heuristics rigorous? Can you apply it? Describe the exceptional case where the heuristic argument cannot be proved rigorously.

24. **Solving the Kepler Problem** Continuing the previous problem, express E and L in terms of polar coordinates (r, θ) and their time derivatives, where $x = r\cos\theta$, $y = r\sin\theta$. Assume $L \neq 0$.

Show that θ is a strictly increasing or strictly decreasing function of time. Rather than considering r as a function of t, consider r as a function of θ. (It is convenient to distinguish $\dot{r} = dr/dt$ from $r' = dr/d\theta$.) Transform both the energy conservation and the ODE for $\ddot{r}$ into ODEs for functions of θ, rather than t. Hindsight then invents the lucky substitution $r = 1/\rho$ to obtain a significant simplication. Solve the ODE for $\rho = \rho(\theta)$.

Finally set up the differential equation connecting θ and t. (We'll omit solving it.)

25. **A Geometric Look at the Kepler Problem** It seems difficult to visualize the set

$$\Lambda(E_0, L_0) := \left\{ (x, y, v, w) \;\middle|\; (x, y) \neq (0, 0)\,,\ \frac{v^2 + w^2}{2} - \frac{1}{\sqrt{x^2 + y^2}} = E_0\,,\ xw - yv = L_0 \right\} \subset \mathbb{R}^4$$

geometrically, doesn't it? — But using the expressions for E and L in polar coordinates, you can visualize an $\mathbb{R}^3$ whose 'horizontal' plane has cartesian coordinates (x, y) or polar coordinates (r, θ), and whose vertical axis denotes $\dot{r}$. ($\dot{\theta}$ is eliminated in favor of $L = L_0$ already.) What kind of geometric (topological) shape does the level set $\Lambda(E_0, L_0)$ have? E.g., is it a (possibly deformed) sphere, or a (possibly deformed) donut, or what else? How does the answer depend on $\operatorname{sign} E_0$? What happens if $E = -1/(2L^2)$?

26. What the Hedgehog Knows about Planetary Motion ; –) *The 'Hedgehog Theorem' from algebraic topology says that you cannot comb a hedgehog. More precisely, it is not possible on a sphere (round, or deformed by a C^1 map with C^1 inverse) to have a continouous assignment of a nonzero tangential vector to each point. In meteorology, it says that, unless there is a tornado somewhere (where the wind is not tangential to the surface), there must be some point on the surface of the earth where there is no wind.*

Assuming this result, what (negative) conclusion could you have drawn on the shape of $\Lambda(E_0, L_0)$ in the previous problem *with virtually no calculation at all*?

Further and much deeper reasoning along this school of thought could even have positively predicted the topological shape of $\Lambda(E_0, L_0)$ instead of just ruling out one possibility. This would transgress the scope of a M531/M532; I just mention it to illustrate the power of a qualitative analysis, even in the absence of explicit calculations, in particular in low dimensions. Compare with the ultra-simple problem 17 for another instance of qualitative reasoning without calculation.

27. Mathematical Pendulum with Friction For the pendulum $\ddot{x} + \varepsilon\dot{x} + \sin x = 0$ with $\varepsilon > 0$ consider the total *mechanical* energy $E = \frac{1}{2}v^2 + 1 - \cos x$ with $v = \dot{x}$. Show: $(x, v) = (0, 0)$ is asymptotically stable, and in particular if $E(x(0), v(0)) < 2$ and $|x(0)| < \pi$, then $x(t) \to 0$ as $t \to \infty$.

Show asymptotic stability also by means of linearization and compare advantages / disadvantages of the two methods.

28. Stability for Lotka–Volterra Consider the equilibrium

$$\begin{array}{l} R_* = \frac{af+bc}{ef+bd} \\ F_* = \frac{ad-ec}{ef+bd} \end{array} \qquad \text{of the system} \qquad \begin{array}{l} \dot{R} = aR - bRF - eR^2 \\ \dot{F} = -cF + dRF - fF^2 \end{array}$$

in the case where this equilibrium is biologically meaningful ($ad > ec$). Show by linearization that the equilibrium is asymptotically stable, provided $a, b, c, d, e, f > 0$.

It is possible to concoct a Lyapunov function of the form $V = \alpha R - \beta \ln R + \gamma F - \delta \ln F + const.$ for the same purpose, but the calculation that determines the appropriate choice of $\alpha, \beta, \gamma, \delta$ is rather messy. With Mathematica's help I obtain
$\alpha = d(ef + bd)$, $\beta = d(af + bc)$, $\gamma = b(ef + bd)$, $\delta = b(ad - ec)$.
I then get $\dot{V} = (ef + bd)[-de(R - R_*)^2 - bf(F - F_*)^2]$.

If you trust my calculation, can you answer the question: Still assuming $a, b, c, d, e, f > 0$, do there still exist non-constant periodic orbits as was the case for $e = f = 0$? Explain.

29. Attractive and Unstable Consider the ODE

$$\begin{array}{l} \dot{x} = x(1 - r^2) + z^2 - r^2 z \\ \dot{z} = z(1 - r^2) + x(r^2 - z) \end{array} \qquad (\text{where } r^2 = x^2 + z^2)$$

(a) Using the auxiliary function $V(x,z) = x^2 + z^2$, calculate $\frac{d}{dt}V$; show that the sets $\{(x,z) \mid V(x,z) \le c\}$ are forward invariant, provided $c \ge 1$. Show that the flow exists for all positive times. Which level sets of V are invariant?

(b) Show that there are exactly two equilibria: $(0,0)$ and one other, which we shall call (x_*, z_*). Re-using the calculation of $\frac{d}{dt}V$ narrows down this calculation significantly.

(c) For each equilibrium, find the linearization of the vector field there, and determine its eigenvalues. What conclusions (if any) can be drawn about the stability of either equilibrium?

(d) Show that (x_*, z_*) is unstable.

(e) Now (begin to) show that (x_*, z_*) attracts all initial values $(x_0, z_0) \ne (0,0)$: to this end, show first that $\lim_{t\to\infty} V(\phi^t(x_0, z_0)) = 1$. Also show that for all initial values (x_0, z_0) that are on the circle $\{V = 1\}$ already, $\lim_{t\to\infty} \phi^t(x_0, z_0) = (x_*, z_*)$.

(f) Use the two facts from (e) together to show that the ω-limit set of any initial point other than $(0,0)$ is either $\{(0,1)\}$ or the entire circle $x^2 + z^2 = 1$.

In which direction does the vector field point on the following line segments or arcs: (1) $z = 1$, $0 < x < 1$; (2) $x^2 + y^2 = 1.9$; (3) $x = 0$, $0 < z < 1$; (4) $z = 0$, $x > 0$. — Use this to identify a forward invariant set and to rule out the entire circle $x^2 + z^2 = 1$ as an ω-limit set.

30. Lorenz Equations: Equilibria and Their Stability The Lorenz equations are known for rather complicated dynamics. They are

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned} \tag{3}$$

with parameters $\beta, \rho, \sigma > 0$,

(a) Show that for $\rho < 1$, the only equilibrium is $(0,0,0)$, and it is asymptotically stable, and that this equilibrium becomes unstable for $\rho > 1$.

(b) Show that for $\rho > 1$, other equilibria occur, and that they are asymptotically stable at least for $\rho \in \,]1, \hat{\rho}[$ with some $\hat{\rho} > 1$ dependent on β, σ.

(c) By considering for which value of ρ (given $\beta, \sigma > 0$) a pair of eigenvalues on the imaginary axis is possible, show that we can take $\hat{\rho} = \dfrac{\sigma^2 + 3\sigma + \sigma\beta}{\sigma - 1 - \beta}$, provided $\sigma > \beta + 1$, and $\hat{\rho} = \infty$ otherwise.

(d) Show that indeed, for $\rho > \hat{\rho}$, the non-trivial equilibria are unstable. To this end, consider β, σ fixed and calculate the derivative $d\lambda_1/d\rho$ of the *negative* eigenvalue when the other eigenvalues are crossing the imaginary axis. (Thanks to a trace argument, this calculation, which is simpler than the derivative of the other eigenvalues, suffices.)

Note: Eigenvalues of a matrix depend continuously on the matrix, and as long as they are simple, they depend differentiably on the matrix.

31. Lorenz Equations: The 'Easy' Part of Long Time Behavior For the Lorenz equations, consider $V(x,y,z) := \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2$. Show that there exist positive constants a, b such that $\dot{V} \le -aV + b$.

Hint: Expressing a, b in terms of β, ρ, σ is a bit messy and not really needed. However, if you use, e.g., $z(z-2\rho) \geq \frac{1}{2}(z-2\rho)^2 - 2\rho^2$, then you'll see that $a = \min\{2\sigma, 2, \beta\}$ and $b = 4\sigma\beta\rho^2$ work.

Conclude that solutions exist for all positive times, and that the ω-limit set of each orbit is contained in the ellipsoid $\{(x, y, z) \mid V(x, y, z) \leq b/a\}$ (and therefore compact and non-empty).

32. Associated Matrix Norms
(a) Show that the matrix norm $\|A\| := \max_{|x|=1} |Ax|/|x|$ associated to the vector norm $|x| := \max_i |x_i|$ is given by $|A| = \max_i \sum_j |A_{ij}|$. *Hint: Do it by means of explicit inequalities $|Ax| \leq$ etc. and with examples where the inequality is sharp. Calculus is no good here since the set $|x| = 1$ is not smooth.*
(b) Show that the matrix norm associated to the vector norm $|x| := (\sum_i |x_i|^2)^{1/2}$ is the square root of the largest eigenvalue of the symmetric matrix $A^T A$. *Hint: Do it with Calculus and Lagrange multipliers; no explicit inequalities.*

33. Explicit Matrix Exponentials
(a) Calculate $\exp \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$ by explicit use of the power series.

(b) Likewise, calculate $\exp \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ and also $\exp \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(c) Confirm explicitly that $\exp\left(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -t & 0 \end{bmatrix}\right)$ is in general

neither $\left(\exp \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right)\left(\exp \begin{bmatrix} 0 & 0 \\ -t & 0 \end{bmatrix}\right)$ nor $\left(\exp \begin{bmatrix} 0 & 0 \\ -t & 0 \end{bmatrix}\right)\left(\exp \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right)$.

34. Strict Lyapunov Functions May Still Look Funny: Let $\alpha > 0$ and $|\beta| < 1$. Transform the system $\begin{cases} \dot{x} = -\alpha x - y \\ \dot{y} = x - \alpha y \end{cases}$ into polar coordinates. Likewise express the function

$$V(x, y) := (x^2 + y^2)\left[1 + \frac{\beta}{\sqrt{x^2 + y^2}}\left(x \cos \frac{\ln(x^2 + y^2)}{2\alpha} - y \sin \frac{\ln(x^2 + y^2)}{2\alpha}\right)\right]$$

in polar coordinates. Show that V is a strict Lyapunov function for the ODE at the origin. Plot a typical orbit of the ODE and, with the same unit length, graph a radial section of the function V, i.e., graph $r \mapsto V(r \cos \varphi, r \sin \varphi)$ for some choice of φ. (For the graph I suggest $\beta = 3/4$, $\alpha = 1/20$.)

35. A little review problem concerning the flow Review the definition and properties of the flow map, if necessary. Then answer the question: Given the flow map $(t, x) \mapsto \phi(t, x)$, find the vector field f that generated it. (I.e., express f in terms of ϕ).

If you think complicated, then you are thinking too complicated...

36.

37.

38.

39.