DO ALL YOUR WORK AND WRITE ALL YOUR ANSWERS ON THE ADDITIONAL PAPER THAT IS PROVIDED. YOU MUST SHOW YOUR WORK IN ORDER TO RECEIVE FULL CREDIT. PLEASE USE ONLY ONE SIDE OF EACH SHEET AND WRITE YOUR NAME ON EACH SHEET. SHOW AS MUCH WORK AS POSSIBLE BECAUSE YOU MAY RECEIVE PARTIAL CREDIT FOR THE WORK YOU DO IF YOUR ANSWER IS INCORRECT. CIRCLE YOUR ANSWERS.

1. The following two models describe the growth of two different populations:

$$a_n = \frac{200n}{n+100} \qquad\qquad b_{n+1} = \frac{200b_n}{b_n+100}, b_0 = 1$$

   a. Find $\lim_{n\to\infty} a_n$.
   b. Find $\lim_{n\to\infty} b_n$.
   c. Find the other equilibrium for b_n. Is it stable or unstable?
   d. What are the differences between these two models?
   e. Which one is more realistic/useful?

BONUS 1: Find an explicit formula for b_n. Hint: Let $b_n = \frac{1}{x_n}$ and find an explicit formula for x_n.

2. 5% of cases of a rare disease are cured each year. If 20 new cases are diagnosed each year:
   a. What will be the equilibrium number of cases?
   b. If the goal is to reduce the equilibrium number of cases to 100, what percentage of cases will need to be cured each year?

3. Populations of trout in two different lakes grow each week according to the following two models:

$$\Delta x = 0.01(1000 - x_n) \qquad\qquad \Delta y = 0.01 y_n$$

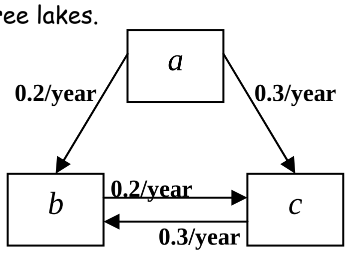
   a. Explain what each of these models means and explain the difference(s) between them.
   b. Rewrite both models as linear difference equations (in standard form).

   The owner of the lakes stocks each lake with 200 trout initially and decides to start allowing fishing in each lake when the population of trout in the lake reaches 600.

   c. For each lake, how many weeks will it take to reach 600 trout?

BONUS 2: For each lake, after reaching 600 trout, what is the maximum number of trout that can be caught each week without driving the population to extinction?

4. 1000 gallons of oil are accidentally released from a wrecked ship into the northernmost lake (Lake *a*) of a three-lake system. After the accident, no oil enters or leaves the system, and there was no oil in any of the lakes before the accident. The diagram below shows the yearly transfer rates of the oil between the three lakes.



   a. Construct a transfer matrix for the system.
   b. How many gallons of oil will be in each lake after 2 years?
   c. How many gallons of oil will be in each lake in the long run (that is, at equilibrium)?

5. Many years ago, a non-native frog population invaded an island and has been growing each year according to the following Leslie matrix: $L = \begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix}$

   a. Explain what each element of the Leslie matrix means.
   b. Find L^2.
   c. Notice that $L^2 = kL$ where k is some scalar number. What number is k?
   d. Find all of the eigenvalues of L and their corresponding <u>normalized</u> eigenvectors.

   The age distribution of the frog population is now at equilibrium, and the total number of female frogs in the population is 10,000.

   e. How many female frogs are in each age class?
   f. What will be the total number of female frogs next year?

BONUS 3: What is the minimum number of female frogs that must be killed in each age class each year in order to eventually drive the population to extinction?

6. Accumulation of mercury in the human body is a serious health concern because it takes the body such a long time to eliminate mercury, and high levels of mercury can damage the nervous system and cause birth defects. Mercury is eliminated from the body exponentially with a half-life of 53 days. The EPA recommends a maximum intake of 0.1 μg of mercury per kg of body weight per day.

   a. What fraction of mercury is eliminated from the body each day?
   b. What is the maximum recommended daily intake of mercury for a 65 kg person?
   c. Write a linear difference equation (in standard form) for the amount of mercury in the body. (Include the information from part a and part b.)
   d. For a 65 kg person that consumes mercury in food at the maximum recommended rate, what will be the equilibrium level of mercury in the body?

   The equilibrium level from part d is also called the maximum safe level of mercury in the body. At the doctor's office, a 65 kg female patient is complaining of symptoms that sound like mercury poisoning. During the doctor's interview, the patient explains that she eats a package of tuna every day for lunch. A blood test of the patient reveals that the level of mercury in her body is $1\frac{1}{2}$ times the maximum safe level, which means that her daily intake of mercury exceeds the EPA recommendation.

   e. How much mercury is in the patient's body?
   f. Write a linear difference equation (in standard form) for the amount of mercury in the patient's body. (NOTE: We don't yet know how much mercury she intakes each day.)
   g. Assuming the amount of mercury in the patient's body is at equilibrium and assuming the tuna is her only source of mercury, how much mercury is she getting from each package of tuna?
   h. If the patient stops eating tuna altogether, how long will it take for the level of mercury in her body to return to the maximum safe level?

BONUS 4: If instead the patient only eats a package of tuna every other day, how long will it take for the level of mercury in her body to return to the maximum safe level?

ANSWERS TO THE REGULAR PROBLEMS:

1.

a. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{200n}{n+100} = \lim_{n\to\infty} \frac{200}{1+\frac{100}{n}} = \frac{200}{1+0} = 200$

b. First, we need to find the equilibria of b_n:

$$b = \frac{200b}{b+100} \Rightarrow b(b+100) = 200b \Rightarrow b^2 + 100b = 200b$$
$$\Rightarrow b^2 - 100b = 0 \Rightarrow b(b-100) = 0$$

So the equilibria are $b=0$ and $b=100$.
Using $b_0 = 1$ and iterating the difference equation several times, we can see that $b_n \to 100$ as $n \to \infty$. So $\lim_{n\to\infty} b_n = 100$.

c. From part b, the other equilibrium is $b=0$ and it is unstable.

d. a_n uses an explicit formula whereas b_n uses a recursive equation (difference equation).
a_n has a single limit whereas b_n has multiple equilibria.
The values of a_n are determined by n and cannot change whereas the values of b_n are determined by b_0, which can change.

e. Because the values of b_n can change by using different values of b_0, it is more flexible and thus more realistic/useful. Also, it handles the case where $b_n = 0$ (meaning that the population size is always zero), which is not possible with a_n.
b_n is an example of the Beverton–Holt stock-recruitment model (Beverton and Holt, 1957), which is used to model fisheries populations.

2. This problem is very similar to homework problem #30 in Section 48 (pg. 542).

a. This problem can be modeled with the following linear difference equation:

$$x_{n+1} = 0.95x_n + 20$$

where 0.95 refers to the 95% of cases that remain uncured each year and 20 refers to the 20 new cases each year. The equilibrium for this model is:

$$x_{eq} = \frac{20}{1-0.95} = \frac{20}{0.05} = 400 \text{ cases}$$

b. To reduce the equilibrium number of cases to 100, we need to solve this equation:

$$x_{eq} = \frac{20}{1-a} = 100 \Rightarrow 20 = 100(1-a) \Rightarrow 0.2 = 1-a \Rightarrow a = 0.8$$

So we need to reduce the percentage of uncured cases to 80% each year, or in other words, we need to cure 20% of cases each year.

3.

a. For Lake *x*, the model means that the change in the population size each week is proportional to the difference between 1000 and the current population size. This is the same model that is used in homework problem #25 in Section 48 (pg. 541). 1000 can be considered the *carrying capacity* of the population.
For Lake *y*, the model means that the change in the population size each week is proportional to the current population size. In other words, the population is growing by 1% each week.

b. $\Delta x = 0.01(1000 - x_n) \Rightarrow x_{n+1} - x_n = 10 - 0.01x_n \Rightarrow x_{n+1} = 10 - 0.01x_n + x_n$

$\Rightarrow x_{n+1} = 0.99x_n + 10$

$\Delta y = 0.01y_n \Rightarrow y_{n+1} - y_n = 0.01y_n \Rightarrow y_{n+1} = 0.01y_n + y_n$

$\Rightarrow y_{n+1} = 1.01y_n$

c. To solve this problem, we need to use the explicit solutions for x_n and y_n:

$$x_{eq} = \frac{10}{1-0.99} = \frac{10}{0.01} = 1000$$

$$x_n = \left(x_0 - x_{eq}\right)\left(0.01\right)^n + x_{eq}$$

$$\Rightarrow 600 = \left(200 - 1000\right)\left(0.99\right)^n + 1000$$

$$\Rightarrow 400 = \left(800\right)\left(0.99\right)^n \Rightarrow \left(0.99\right)^n = \tfrac{400}{800} = 0.5$$

$$\Rightarrow \ln\left(0.99\right)^n = \ln\left(0.5\right) \Rightarrow n\ln\left(0.99\right) = \ln\left(0.5\right)$$

$$\Rightarrow n = \frac{\ln\left(0.5\right)}{\ln\left(0.99\right)} \approx 69$$

$$y_n = y_0\left(1.01\right)^n$$

$$\Rightarrow 600 = 200\left(1.01\right)^n \Rightarrow \left(1.01\right)^n = \tfrac{600}{300} = 3$$

$$\Rightarrow \ln\left(1.01\right)^n = \ln\left(3\right) \Rightarrow n\ln\left(1.01\right) = \ln\left(3\right)$$

$$\Rightarrow n = \frac{\ln\left(3\right)}{\ln\left(1.01\right)} \approx 110.4$$

So it will take 69 weeks for the trout population in Lake *x* to reach 600 and it will take 111 for the trout population in Lake *y*.

4.

a. $T = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0.3 \\ 0.3 & 0.2 & 0.7 \end{bmatrix}$

b. $\mathbf{x}_1 = T\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0.3 \\ 0.3 & 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 500 \\ 200 \\ 300 \end{bmatrix}$

$\mathbf{x}_2 = T\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0.3 \\ 0.3 & 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 500 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 250 \\ 350 \\ 400 \end{bmatrix}$

So there will be 250 gallons in Lake *a*, 350 gallons in Lake *b*, and 400 gallons in Lake *c*.

c. At equilibrium, we can find the normalized eigenvector as follows:

$$T\mathbf{x} = \mathbf{x} \Rightarrow \begin{bmatrix} 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0.3 \\ 0.3 & 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} 0.5a \\ 0.2a + 0.8b + 0.3c \\ 0.3a + 0.2b + 0.7c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{matrix} & 0.5a = a & & a = 0 & & a = 0 \\ \Rightarrow & 0.2a + 0.8b + 0.3c = b & \Rightarrow & 0.3c = 0.2b & \Rightarrow & 1.5c = b \\ & 0.3a + 0.2b + 0.7c = c & & 0.2b = 0.3c & & b = 1.5c \end{matrix}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 1.5 \\ 1 \end{bmatrix} \div 2.5 = \begin{bmatrix} 0 \\ 0.6 \\ 0.4 \end{bmatrix}$$

This means that at equilibrium, 0% of the total oil (0 gallons) will be in Lake *a*, 60% of the total oil (600 gallons) will be in Lake *b*, and 40% of the total oil (400 gallons) will be in Lake *c*.

5.

a. Juvenile females give birth to an average of 0.8 juvenile females each year. Adult females give birth to an average of 1.6 juvenile females each year. 20% of juvenile females survive to adulthood each year. (80% die each year.) 40% of adult females survive from year-to-year. (60% die each year.)

b. $L^2 = L \cdot L$

$$L^2 = \begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.8\cdot 0.8 + 1.6\cdot 0.2 & 0.8\cdot 1.6 + 1.6\cdot 0.4 \\ 0.2\cdot 0.8 + 0.4\cdot 0.2 & 0.2\cdot 1.6 + 0.4\cdot 0.4 \end{bmatrix}$$

$$L^2 = \begin{bmatrix} 0.96 & 1.92 \\ 0.24 & 0.48 \end{bmatrix}$$

c. $k = 1.2$ since

$$L^2 = \begin{bmatrix} 0.96 & 1.92 \\ 0.24 & 0.48 \end{bmatrix} = \begin{bmatrix} 1.2\cdot 0.8 & 1.2\cdot 1.6 \\ 1.2\cdot 0.2 & 1.2\cdot 0.4 \end{bmatrix} = 1.2\begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix} = 1.2L$$

d. For the eigenvalues:

$$0 = \lambda^2 - tr(L)\lambda + \det(L) = \lambda^2 - (0.8 + 0.4)\lambda + (0.8\cdot 0.4 - 0.2\cdot 1.6)$$

$$0 = \lambda^2 - 1.2\lambda = \lambda(\lambda - 1.2)$$

$$\Rightarrow \lambda_d = 1.2, \lambda_2 = 0$$

So the dominant eigenvalue is 1.2, and the other eigenvalue is 0.

For the normalized eigenvector corresponding to the dominant eigenvalue:

$$L\mathbf{x} = 1.2\mathbf{x} \Rightarrow \begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} j \\ a \end{bmatrix} = 1.2\begin{bmatrix} j \\ a \end{bmatrix} \Rightarrow \begin{bmatrix} 0.8j + 1.6a \\ 0.2j + 0.4a \end{bmatrix} = \begin{bmatrix} 1.2j \\ 1.2a \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 0.8j + 1.6a = 1.2j \\ 0.2j + 0.4a = 1.2a \end{matrix} \Rightarrow \begin{matrix} 1.6a = 0.4j \\ 0.2j = 0.8a \end{matrix} \Rightarrow \begin{matrix} 4a = j \\ j = 4a \end{matrix}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \div 5 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

For the normalized eigenvector corresponding to the other eigenvalue:

$$L\mathbf{x} = 0\mathbf{x} \Rightarrow \begin{bmatrix} 0.8 & 1.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 0.8x + 1.6y \\ 0.2x + 0.4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 0.8x + 1.6y = 0 \\ 0.2x + 0.4y = 0 \end{matrix} \Rightarrow \begin{matrix} 0.8x = -1.6y \\ 0.2x = -0.4y \end{matrix} \Rightarrow \begin{matrix} x = -2y \\ x = -2y \end{matrix}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \div 1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

e. At equilibrium, 80% of the female frogs (8,000 frogs) are juveniles, and 20% of the female frogs (2,000 frogs) are adults.

f. Next year there will be 12,000 female frogs (since $12,000 = 1.2 \cdot 10,000$).

6.

a. First we need to find the rate that the mercury is decaying:

$$r = \frac{\ln 2}{t_h} = \frac{\ln 2}{53} \approx 0.013$$

Then the fraction that remains in the body after one day is:

$$N(1) = N_0 e^{-r \cdot 1} = N_0 e^{-0.013} \approx 0.987 N_0$$

which means that 98.7% of the mercury remains in the body each day, or in other words, 1.3% is eliminated from the body each day.

b. The maximum recommended daily intake of mercury for a 65 kg person is 6.5 μg.

$$65kg \cdot 0.1 \,{}^{\mu g}\!/_{kg} = 6.5 \mu g$$

c. $x_{n+1} = 0.987 x_n + 6.5$

d. The equilibrium level of mercury in the body is 500 μg.

$$x_{eq} = \frac{6.5}{1-0.987} = \frac{6.5}{0.013} = 500$$

e. There are 750 μg of mercury in the patient's body.

f. $x_{n+1} = 0.987 x_n + b$ (where b stands for the amount of mercury that the patients intakes each day)

g. The patient is getting 9.75 μg of mercury from each package of tuna, which is 1½ times the maximum recommended daily intake.

$$x_{eq} = \frac{b}{1-0.987} = 750 \Rightarrow b = 750 \cdot 0.013 = 9.75$$

h. If the patient stops eating tuna altogether, the linear difference equation for the amount of mercury in her body becomes:

$$x_{n+1} = 0.987 x_n$$

To find out how long it takes, we need to use the explicit solution for x_n:

$$x_n = x_0 (0.987)^n$$

$$\Rightarrow 500 = 750 (0.987)^n \Rightarrow (0.987)^n = \tfrac{500}{750} = \tfrac{2}{3}$$

$$\Rightarrow \ln (0.987)^n = \ln\left(\tfrac{2}{3}\right) \Rightarrow n \ln(0.987) = \ln\left(\tfrac{2}{3}\right)$$

$$\Rightarrow n = \frac{\ln\left(\frac{2}{3}\right)}{\ln(0.987)} \approx 31$$

So it will take 31 days for the level of mercury in her body to return to the maximum safe level.