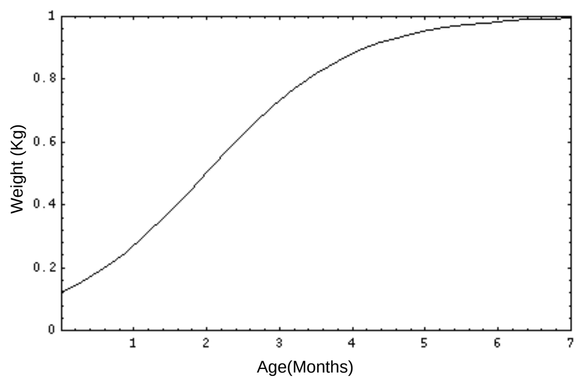
Math 152 - Sample Final Exam - Spring 2005

1. As a laboratory assignment in a biology class, you are asked to make observations on a fungal culture in a petri dish to test the hypothesis that the growth rate of the culture is proportional to the culture's size. How could you do this assuming that the easiest observation you can make is the area of the fungal colony? Give a condition under which you would conclude this hypothesis is refuted.

2. The growth in weight of a fish is given by



If W(t) gives the fish weight on month t, (a) Define W'(t) and state its units. (b) Estimate W'(2). (c) Sketch a graph of how (approximately) W'(t) changes from age zero to age 6 months.

3. Find the following:

(a) $D_t\ y(t)$ if $y(t) = 4\, t \ln(2t+1)$ (b) $g'(x)$ if $g(x) = x/(x\ \ +1)$

(c) $\dfrac{df}{dy}$ if $f(y) = \dfrac{1}{(3y^2 + 2y)^3}$ (d) $D_t y(t)$ if $y(t) = e^{\cos(4t)}$

4. The above ground biomass of a uniform age stand of trees is given by

$$B(a) = \frac{1500}{10 + 90\, e^{-a/10}}$$ tons/hectare where a is the stand age in years.

(a) If K = long-term biomass in this stand, find K. (b) What is the growth rate of the stand biomass? (c) At what age will the stand biomass be 1/2 of the long-term biomass (e.g. find a so that B(a) = K/2). (d) It can be shown that

$$B'(a) = r\, B\left(\frac{K - B}{K}\right)$$ where r = 1/10 and K is the value found in (a)

Show that B′(a) is maximized when B(a) = K/2 and thus state in words why you might wish to harvest the stand at the age you found in (c)

5. Find N(t) if $N''(t) = 3\,t + 1$ and $N'(0) = 4$ and $N(0) = 2$

6. Find the area bounded between the graphs of $y = 2\,x^2$ and $y = 4x$.

7. Find the following:

(a) $\int (3 x e^{-4x})\, dx$ (b) $\int_{-1}^{1} \frac{1}{(2-x)^3}\, dx$ (c) $\int \frac{\ln x}{x}\, dx$

8. The length of a fish of age t grows according to
$L'(t) = K (L_\infty - L)$ so $L(t) = L_\infty(1 - e^{-K(t-t_0)})$
with $L_\infty = 40$ cm, K = .2/month and L(0)=3 cm. (a) How fast is the fish growing in length at birth? (b) At what age will the fish reach 1/2 of its largest length?

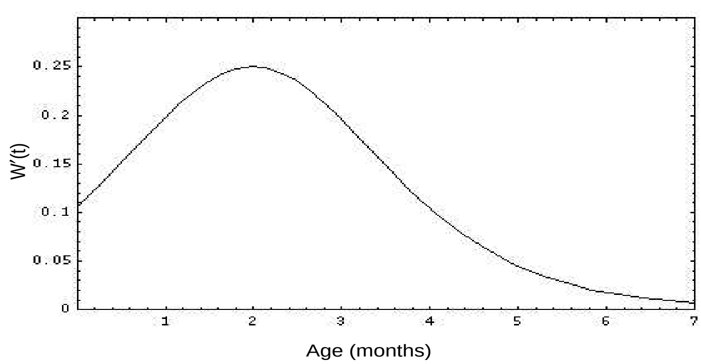
9. How much work is necessary to pump the contents of a tank of liquid of density 40 kg per cubic meter just over the top of the tank if the tank is cone shaped with a circular base of radius 6 meters and a circular top of radius 3 meters with the top being 12 meters above the ground. The tank is half full of liquid at the start (e.g. the level of liquid in the tank is 6 meters above the bottom of the tank). .

10. Find (a) the solution of $y'=(2t+1)\, y$ if y(0)=4.
(b) all solutions of $\frac{dN}{dt} = \frac{N}{2t}$

Answers

1.If A(t) = area of fungal culture at time t, A'(t)=kA(t) implies $A(t)=A(0)\, e^{kt}$.
So measure A(t) at several times (e.g. $t_1, t_2, t_3, \ldots$) and since lnA(t) = kt+lnA(0), plot t_i versus $A(t_i)$ on semilog paper (equivalent to plotting t_i versus $\ln A(t_i)$ on regular graph paper). If this gives a linear graph, accept the hypothesis, otherwise reject it. (Note: you could calculate R^2 from the linear regression and reject the hypothesis if R^2 is not above .5 say)

2. By Definition $W'(t) = \lim_{h\to 0} \frac{W(t+h) - W(t)}{h}$ gives the growth rate of the fish at a particular instant t, in kg/month and W'(2) is approximately .25 kg/month



3. (a) $y'(t) = 4 \ln (2t+1) + 8t /(2t+1)$ (b) $g' = \frac{1}{(x + 1)^2}$ (c) $f' = \frac{-6\,(3y+1)}{(3y^2 + 2y)^4}$

(d) $y'(t) = -4 (\sin 4t)\, e^{\cos 4t}$

4. (a) K = 150 tons/hectare (b) $B'(a) = \dfrac{13500\ e^{-a/10}}{(10 + 90\ e^{-a/10})^2}$

(c) B(a) = 75 $\Rightarrow$ a = 10 ln9 =22.0 years

(d) $B''(a) = r B' \left(\dfrac{K - B}{K}\right) + r B\left(-\dfrac{B'}{K}\right) = r B'\left(1 - \dfrac{2B}{K}\right) = 0$ when B = K/2 so B′ is maximized when B=K/2. This implies that the stand being harvested at 22 years would be harvested when the biomass growth rate has been maximized. Harvesting later than this would give a period of stand growth at lower than the maximum growth rate.

5. $N(t) = \dfrac{1}{2} t^3 + \dfrac{t^2}{2} + 4 t + 2$

6. $\displaystyle\int_0^2 (4 x - 2 x^2)\ dx = 8/3$

7. (a) $-\dfrac{3}{4} e^{-4x}\left(x + \dfrac{1}{4}\right) + C$ (b) $\dfrac{4}{9} = .44$ (c) $\dfrac{(\ln x)^2}{2} + C$

8. (a) L′(0) = .2 (40 - 3) = 7.4 cm/month

(b) $L(0) = 3 = 40(1 - e^{.2 t_0}) \Rightarrow t_0 = 5 \ln(37/40) = -.39$

so $L(t) = 20 = 40 (1 - e^{-.2(t + .39)})$ which implies t = -5 ln(1/2)-.39 = 3.07 months

9. $$\int_0^6 40\ (12 - x)\ \pi \left(\frac{-1}{4} x + 6\right)^2 dx = 61830\ \pi = 194244 \text{ kg m}$$

10.(a) $y(t) = 4 e^{t^2 + t}$ (b) $N(t) = c\ t^{1/2}$