Approximating the entropy of a 2-dimensional shift of finite type

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Abstract. In this paper, we extend the method used to compute entropy of 1-dimensional subshift and the technique by Calkin and Wilf to 2-dimensional case. This allows us to compute entropies easily for many unknown cases, some of which will be discussed in the last section.

1. Introduction. In symbolic dynamics, we study a collection of sequence of symbols with a certain restriction. We let $\mathcal{A}$ be a finite set of symbols. Each element of $\mathcal{A}$ is called a symbol, or an *alphabet*, usually denoted by $a, b, c, \ldots$ or $0, 1, 2, \ldots$.

Definition 1-1. The *full $\mathcal{A}$-shift*, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of alphabets from $\mathcal{A}$. If the set of alphabet is understood, we could say the full shift instead of the full $\mathcal{A}$-shift.

Formally,

$$\mathcal{A}^{\mathbb{Z}} = \left\{ \mathbf{x} = (x_i)_{i \in \mathbb{Z}} | x_i \in \mathcal{A},\, \forall i \in \mathbb{Z} \right\}$$

Each sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is called a *point*. We write $x = (x_i)_{i \in \mathbb{Z}}$, where each x_i is an element of $\mathcal{A}$ and is indexed through integers.

A *block*, or *word* is a finite string(i.e. finite consecutive symbols) of a sequnce in $\mathcal{A}^{\mathcal{Z}}$. A *length* of a block $\mathbf{u}$ is a number of symbols in $\mathbf{u}$. We also call a block with length n by *n-block*.

Next, we introduce a notion of a shift space, which is a main object of study in symbolic dynamics.

Definition 1-2. Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$. A *shift space* or *subshift* X is a subset of sequences in the full shift which do not contain any blocks in $\mathcal{F}$. We call $\mathcal{F}$ a *forbidden blocks* and denote a shift space with a forbidden block $\mathcal{F}$ by $X_{\mathcal{F}}$.

For a shift space X, we define *allowed n-blocks* to be a set of all n-blocks appearing in X. We denote the set by $B(n)$ and its number of elements by B_n. For a subshift $X_{\mathcal{F}}$, we easily see that $B(n)$ is a collection of all n-blocks which do not contain any block in $\mathcal{F}$.

We note that different sets of forbidden blocks could give the same subshift. For example, if we let $\mathcal{F}_1 = \{000\}$ and $\mathcal{F}_2 = \{0001, 0000\}$, then we have $X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$ as a shift space of $\{0,1\}^{\mathcal{Z}}$. We can construct infinitely many numbers of sets of forbidden blocks by extending a certain forbidden block to blocks of fixed size containing it. However, we mostly prefer the forbidden set with less elements.

The most important class of shift spaces is a *shift of finite type* or in short *SFT* , a shift space which is determined by a finite numbers of forbidden blocks. Consequently, we classify shifts of finite types by their element with greatest length.

Definition 1-3. A shift of finite type is *m-step* if the longest block of its forbidden set is of length m+1. From the above discussion, we see that if X is an m-step shift of finite type, then it is all so k-step for $k \geq m$. An m-step shift of finite type $X_{\mathcal{F}}$ is completely determined by m-blocks, that is we can check whether a point x is in $X_{\mathcal{F}}$ by considering if all (m+1)-blocks of x is allowed [**1**].

Example 1-4. The *golden mean shift* is a subshift $X_{\mathcal{F}}$ in $\{0,1\}^{\mathbb{Z}}$ where $\mathcal{F} = \{11\}$. In other words, it is a collection of all bi-infinite strings of 0,1 containing no consecutive 1's. This shift is 1-step and the allowed 3-blocks are 000, 001, 010, 100, 101. In particular, it is named *the golden mean shift*, because B_n turns out to be a Fibonacci sequnce and involves the golden mean.

A 1-step shift has a property that it could be represented by a direct graph. Each alphabet is represented by a vertice. For the vertices **i** and **j** representing alphabets i and j respectively, there is an edge from **i** to **j** if and only if i can be followed by j (i.e. ij is an allowed block). This makes sense because the only matter for constructing a point for a 1-step shift is whether which alphabet can follow any particular alphabet. In this way, a bi-infinite path of a graph represents a point in a shift while a finite path represents a block. This type of representation is called a vertex shift.

Example 1-5. The golden mean shift can be represented by a graph with 2 vertices. There will be edges form 0 to 1 and 1 to 0, and also a self loop at 0 as Fig. 1-1. There is no self loop at 1 since the block 11 is prohibited.

0 1

Figure 1-1. Graph representation of the golden mean shift.

Note that a certain shift space can have several graph representation. There is also an algorithm to encode a shift of finite type, need not be 1-step, into a graph. There is also a graph with edges, instead of vertices, representing symbols.

Given a graph, we can find an associated *adjacency matrix*, which is a matrix indexed by vertices of a graph. Each entry $A_{I,J}$ is a number of edges from I to J. An adjacency matrix for the graph in Fig. 1-1 is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

An adjacency matrix of a graph representing a subshift leads to a way of calculating a number of allowed n-block.

Proposition 1-6. *Let A be an adjacency matrix of a graph of subshift X. The number of all paths of length m is $\sum_{i,j} A^m$. Consequently, as a sequence of vertices represents an allowed block, $B_{m+1} = \sum_{i,j} A^m$*

This connection plays an important role in the study of symbolic dynamics as many results in matrix theory come to help.

2. Entropy and Perron-Frobenius theorem. An entropy, or sometimes called *complexity*, of a shift space is an important quantification showing a growth rate of allowed blocks over their length. In general, the entropy can be defined for general dynamical systems, but here we will present a special formulation for symbolic dynamics.

Definition 2-1. For a shift space X, we define the entropy, $h(X)$ to be,

$$h(X) = \lim_{n\to\infty} \frac{\log B_n}{n}$$

To prove the existence of this limit, we begin with a fundamental inequality for a shift space

Lemma 2-2. *$B_{m+n} \le B_m B_n$ for any $m, n \in \mathbb{Z}^+$*

Proof: Let X be a shift space with a forbidden block $\mathcal{F}$. Consider an allowed (m+n)-block, it does not contain any forbidden block, so any subblock of it contain none of forbidden block either. Consequently, its first m symbols and its last n-symbols form allowed m-block and allowed n-block respectively. Moreover, each allowed (m+n)-block determines a unique pair of its initial m-subblock and last n-subblock. Therefore, the inequality holds. □

Lemma 2-3. *For a sequence (a_n) of nonnegative real numbers satisfying $a_{m+n} \le a_m + a_n$ for all $m, n \in \mathbb{N}$, then we have $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf_{n\ge 1} \frac{a_n}{n}$*

Considering a sequence $(\log B_n)$ and combining these two lemmas, we verify that the definition of the entropy is well-defined.

Example 2-4. Let X be a full shift with an alphabet set $\mathcal{A}$ of size c. Then $B_n = c^n$, and $h(X)$ is simply $\log c$.

Example 2-5. Let X be a golden mean shift. Considering allowed words of length $n+2$, we partition these words to two groups; ones starting with 0 and the others starting with 1. These groups have bijective correspondences with $B(n+1)$ and $B(n)$ respectively. Thus, we have a recurrence relation $B_{n+2} = B_{n+1} + B_n$, given $B_1 = 2$ and $B_2 = 3$. This is actually a fibonacci sequence, so we have a formula

$$B_n = \frac{1}{\sqrt{5}}\left(\alpha^{n+2} - \beta^{n+2}\right)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then,

$$\begin{aligned} h(X) &= \lim_{n\to\infty} \frac{\log B_n}{n} && \text{(2-1)} \\ &= \lim_{n\to\infty} \frac{1}{n}\left[(n+2)\log\alpha + \log\frac{1}{\sqrt{5}} + \log\left(1 - \frac{\beta^{n+2}}{\alpha^{n+2}}\right)\right] && \text{(2-2)} \\ &= \log\alpha = \log\frac{1+\sqrt{5}}{2} && \text{(2-3)} \end{aligned}$$

This is also a reason why the shift is called a golden mean shift.

Next, we present the Perron-Frobenius theorem which is a fundamental result in matrix theory, but also playing an important role in computing entropy of a subshift.

Definition 2-6. A nonnegative matrix A is *irreducible* if for each pair of indices i, j, there is an integer $n \geq 0$ such that $(A^n)_{i,j} > 0$.

Remark. A graph G is irreducible if and only if its adjacency matrix is irreducible.

Theorem 2-7 (Perron-Frobenius Theorem)**.** *Let A be a nonnegative irreducible matrix. Then A has a positive eigenvalue λ_A with a positive eigenvector v_A along with following properties:*

(1) *For any eigenvalue λ of A, then $|\lambda| \leq \lambda_A$. That is, λ_A is the largest eigenvalue of A.*
(2) *λ_A is both geometrically and algebraically simple.*
(3) *Only positive eigenvectors of A positive are scalar multiple of v_A.*

Corollary 2-8 (Eigenvalue estimate)**.** *There are positive constant c_0 and d_0 such that*

$$c_0 \lambda_A^n \leq \sum_{i,j} A^n \leq d_0 \lambda_A^n$$

Note. This largest eigenvalue of the matrix is called *Perron-Frobenius eigenvalue, or* $\lambda_{max}(A)$, as well as Perron-Frobenius eigen vector for the associated eigenvector.

Combining with Proposition 1-6, we have a way to easily compute an entropy for a shift of finite type.

Proposition 2-9. *Let X be a shift of finite type with a graph and adjacency matrix A. Then, $h(X) = \log \lambda_{max}(A)$*

Proof: By the previous corollary, we get

$$c_0 (\lambda_{max}(A))^n \leq \sum_{i,j} A^n = B_{n+1} \leq d_0 (\lambda_{max}(A))^n$$

$$c_0^{1/n+1} (\lambda_{max}(A))^{n/n+1} \leq B_{n+1}^{1/n+1} \leq d_0^{1/n+1} (\lambda_{max}(A))^{n/n+1}$$

By taking logarithm and letting n go to infinity, we obtain $h(X) = \log \lambda_{max}(A)$. □

We finish the section with an example of computing an entropy of the golden mean shift by applying Perron-Frobenius theorem.

Example 2-10. In the previous section, we find eigenvalus of the adjacency matrix of the golden mean shift by solving its characteristic polynomial.

$$P(x) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$$

We solve roots of the polynomial to be $\left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\}$, then the Perron-Frobenius eigenvalue is $\frac{1+\sqrt{5}}{2}$. Hence, the entropy is $\log \frac{1+\sqrt{5}}{2}$, cosistent with the preceding example.

3. 2-dimensional shift space. Instead of a string of symbols, we now consider an infinite array of symbols. Several notions are defined similarly to those of 1-dimensional shift space. However, many problems in 2-dimensional shift space are more difficult to handle.

The full $\mathcal{A}$-shift, denoted by $\mathcal{A}^{\mathbb{Z}^2} = \left\{ \mathbf{x} = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} | x_{\mathbf{n}} \in \mathcal{A},\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\}$

For each point $x = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2}$, here we have $\mathbf{n}$ indexed through $\mathbb{Z}^2$.

The notion of a block is now broader since there is more freedom to choose a subset of an array. We can define a block, or *pattern* to be a subset of symbols of a point over a finite subset of $\mathbb{Z}^2$. The examples of patterns are $\begin{smallmatrix} 1 & & & \\ 1 & 1 & 1 & 1 \end{smallmatrix}$ and $\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}$.

Given a collection of patterns $\mathcal{F}$, we define a shift space, or subshift $X_{\mathcal{F}}$ to be a set of points in $\mathcal{A}^{Z^2}$ which does not contain any pattern in $\mathcal{F}$. Similar to 1-dimensional shift space, $\mathcal{F}$ is called a forbidden pattern and $X_{\mathcal{F}}$ is a shift of finite type if $\mathcal{F}$ is finite. We also define an allowed pattern to be a pattern which contains no pattern in $\mathcal{F}$.

Example 3-1. Let $X_{\mathcal{F}}$ be a subshift with alphabets $\{0, 1\}$ and $\mathcal{F} = \{ 11, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \}$. This shift is named *2-dimensional golden mean shift*, as an analogue of 1-dimensional golden mean shift in the sense that no two consecutive 1's can occur. Despite simplicity, little information of this subshift is known.

In 2-dimensional shift space, we define an *(m×n)-block* to be a pattern of symbols on a rectangle consisting of n consecutive rows and m consecutive columns of symbols. For instance, a block $\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}$ is a (4×3)-block and a block $\begin{smallmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}$ is an allowed (2×4)-block in $X_{\mathcal{F}}$ where $\mathcal{F} = \{ \begin{smallmatrix} 1 & \\ 1 & 1 \end{smallmatrix} \}$. We also call an (m×n)-block has *width* m and *height* n. In addition, we may generalize a notion of width and height for any pattern by defining to be width and height of the smallest rectangle which can cover that pattern (e.g. $\begin{smallmatrix} & & & 1 & \\ 1 & & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \end{smallmatrix}$ has width 5 and height 3). For a given subshift, we denote $B(m, n)$ to be a set of allowed (m×n)-block and $B_{m,n}$ to be its cardinality.

The entropy for a subshift is now a growth rate of a number of allowed patterns on a rectangle, then the formula of the entropy shall be

$$h(X) = \lim_{m,n \to \infty} \frac{\log B_{m,n}}{mn}$$

Many people also define an entropy to be a growth rate over a square, that is

$$h(X) = \limsup_{n \to \infty} \frac{\log B_{n,n}}{n^2}$$

Though the latter formula guarantees the existence of the limit since $B_{n,n}$ is bounded above by $|\mathcal{A}|^{n^2}$, these definitions agree in most cases. For convenince, we define a *hard entropy*, η as an exponential of an entropy, that is

$$\eta(X) = \lim_{m,n \to \infty} B_{m,n}^{1/mn}$$

In contrast to 1-dimensional shift, there is no method to compute an entropy of a shift of finite type in general. Only entropies of a few special classes of subshifts are known explicitly.

4. Transfer matrix method. The transfer matrix method was first invented by Engel to approximate the entropy of 2-dimensional golden mean shift, this constant is also known as the *hard square entropy* as relating to the hard square model in physics. Later on, it was improved by Calkin and Wilf for better numerical result. See [**4**] for more historical remarks of the constant. We now generalize the method to more classes of SFTs.

Definition 4-1. An *(horizontal) n-strip* of 2-dimensional subshift $X_{\mathcal{F}}$ is a 1-dimensional subshift with an alphabet set $B(1,n)$, constrained by patterns in $\mathcal{F}$ with height no more than n. This is clearly a shift of finite type since its forbidden set is a subset of $\mathcal{F}$. Pictorially, this is a 2-dimensional subshift restricted to an infinite strip of finite height, n.

In this paper, it is concured that the n-strip is horizontal. Although, in general, we can define the strip to be vertical as well.

Example 4-2. A 1-strip of the 2-dimensional golden mean shift is a 1-dimensional golden mean shift.

Definition 4-3. A *transfer matrix* of an n-strip T_n, is a square matrix indexed by the set $B(1,n)$ defined as following

$$(T_n)_{I,J} = \begin{cases} 1 & \text{if } I \text{ can be followed on the right by } J, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $(T_n)_{I,J} = 1$ when IJ is an allowed $2 \times n$ word.

Example 4-4. Let X be a golden mean shift. Consider elements of $B(1,3))$, which are $\substack{0\\0\\0}, \substack{0\\0\\1}, \substack{0\\1\\0}, \substack{1\\0\\0}, \substack{1\\0\\1}$, then we get T_3 with rows and columns indexed by this order

$$T_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Again, we can freely choose the direction for transfer matrix(e.g. right-to-left, down-to-top for $n \times 1$-word. In this paper, we will keep the direction from left to right along with a choice of horizontal n-strip. Moreover, if a set of forbidden patterns has a certain symmetric property, then the transfer matrix will be the same for any chosen direction. Next, we present the result by directly applying a technique by Calkin and Wilf.

Proposition 4-5. *Let $X_{\mathcal{F}}$ be a 2-dimensional SFT and T_n be its transfer matrix. Let λ_n be the Perron-Frobenius eigenvalue of T_n and η be a hard entropy of $X_{\mathcal{F}}$. If these conditions hold:*

(1) *The width of each forbidden pattern is not greater than* 2
(2) *T_n is symmetric for every n*
(3) *$B_{m,n} = B_{n,m}$ for every $m, n \in \mathbb{N}$*

, then we have

$$\left(\frac{\lambda_{p+2q}}{\lambda_{2q}}\right)^{1/p} \leq \eta \text{ for any } p, q \in \mathbb{Z}_0$$

Moreover, λ_n is an entropy of the n-strip.

Definition 4-6. The *circular transfer matrix* C_n is the transfer matrix but indexed by only a bolcks of which initial and terminal alphabets are the same.

Example 4-7. For the matrix C_4 of the golden mean shift. It is indexed by $\begin{smallmatrix}0\\0\\0\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}, \begin{smallmatrix}0\\1\\0\\0\end{smallmatrix}, \begin{smallmatrix}1\\0\\0\\1\end{smallmatrix},$ so

$$C_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

We use this inequality to approximate entropies of some SFTs in the last section.

5. Power transfer matrix. In this section, we introduce generalized notion of transfer matrix for subshifts which are not 1-step.

Definition 5-1. We define the *m-th transfer matrix* of n-strip to be $T_{m,n}$ with dimension $B_{1,n} \times B_{1,n}$ indexed by each $1 \times n$ allowed pattern such that

$$(T_{m,n})_{\mathbf{i},\mathbf{j}} = \text{number of } (m+1) \times n\text{-words strating with } \mathbf{i} \text{ ending with } \mathbf{j}$$

Note that

$$\sum_{i,j} (T_{m,n})_{i,j} = B_{m+1,n}$$

and if an n-strip is 1-step, then $T_{m,n} = T_{1,n}^m$

Lemma 5-2. *$T_{m,n}T_{k,n} \geq T_{m+k,n}$ for any m, n, k positive integers.*

Proof: Each entry of $T_{m+k,n}$ is the number of ((m+k+1)×n)-words strating with **i** ending with **j** which is constructed by merging (m+1)×n-words and (k+1)×n-words. Thus this number is less than or equal to merging arbitrary (m+1)×n-words and (k+1)×n-words. □

Consequently, we have $T_{m,n}^k \geq T_{km,n}$ by repeatedly applying the lemma.

Next, we find a similar relation between largest eigenvalues of transfer matrices. Denote $\lambda_{m,n} = \lambda_{max}(T_{m,n})$

Lemma 5-3. *$\lambda_{m,n}^{1/m} \geq \lambda_{km,n}^{1/km}$ for any m, n, k positive integers.*

Proof: We see that each transfer matrix is irreducible because we are considering a subshift of finite type. Since $T_{m,n}^k \geq T_{km,n}$, then we apply Theorem 4.4.7 from [**5**] and that $\lambda_{max}\left(T_{m,n}^k\right) = \lambda_{max}(T_{m,n})^k$. Thus $\lambda_{m,n}^k \geq \lambda_{km,n}$ and the result follows by taking km-th root. □

Let η_n be the hard entropy of an n-strip

Theorem 5-4. *For each m,n positive integer,*

$$\frac{\lambda_{m,n}^{1/m}}{A^{m_0 n/m}} \leq \eta_n \leq \lambda_{m,n}^{1/m}$$

moreover, $\lambda_{m,n} \rightarrow \eta_n$ *as* $m \to \infty$.

Proof: By Perron-Frobenius eigenvalue dominating principle, we have

$$c_{m,n}\lambda_{m,n}^k \leq \sum_{i,j} \left(T_{m,n}^k\right)_{i,j} \leq d_{m,n}\lambda_{m,n}^k$$

for some positive constant $c_{m,n}$ and dm, n. Then, we get

$$B_{km+1,n} = \sum_{i,j} \left(T_{km,n}\right)_{i,j} \leq \sum_{i,j} \left(T_{m,n}^k\right)_{i,j} \leq d_{m,n}\lambda_{m,n}^k$$

$$B_{km+1,n}^{\frac{1}{km+1}} \leq d_{m,n}^{\frac{1}{km+1}} \lambda_{m,n}^{\frac{1}{m+\frac{1}{k}}}$$

By letting k go to infinity, we have

$$\eta_n \leq \lambda_{m,n}^{1/m}$$

For the other direction, we observe that for each word counted by $T_{m,n}^k$, the forbidden word could appear only in the merging region. Since this is a subshift of finite type, we can assume the merging region has a constant width, say m_0.Then for each ((km+1)×n)-word, we replace all $A^{km_0 \times n}$ blocks in the merging region. Hence

$$A^{km_0 \times n} B_{km+1,n} = A^{km_0 \times n} \sum_{i,j} \left(T_{km,n}\right)_{i,j} \geq \sum_{i,j} \left(T_{m,n}^k\right)_{i,j} \geq c_{m,n}\lambda_{m,n}^k$$

$$A^{\frac{nm_0}{m+\frac{1}{k}}} B_{km+1,n}^{\frac{1}{km+1}} \geq c_{m,n}^{\frac{1}{km+1}} \lambda_{m,n}^{\frac{1}{m+\frac{1}{k}}}$$

The result follows in a similar way. □

We now establish a connection between the entropy of n-strip and the entropy of the subshift.

Theorem 5-5. *Let* η *be the hard entropy of the subshift, then for any positive integer* n*,*

$$\frac{\eta_n^{1/n}}{A^{m_0/n}} \leq \eta \leq \eta_n^{1/n}$$

Proof: By lemma 2-2, we have

$$B_{m,nk} \leq B_{m,n}^k$$

$$B_{m,nk}^{1/m} \leq \left(B_{m,n}^{1/m}\right)^k$$

By letting m go to infinity, we get

$$\eta_{nk} \leq \eta_n^k$$

$$\eta_{nk}^{1/nk} \leq \eta_n^{1/n}$$

By letting k go to infinity, we obtain

$$\eta \leq \eta_n^{1/n}$$

When we concatenate k blocks, the forbidden block could occur in the concatenating region. Then, for each good (m×nk)-block, we construct new (m×nk)-blocks by replacing each symbol in the concatenating region by any alphabet. A number of (m×nk)-blocks constructed by concatenating k good (m×n)-blocks must be less than a number of these new blocks, that is

$$B_{m,n}^k \leq A^{km_0 \times m} B_{m,nk}$$

$$\left(B_{m,n}^{1/m}\right)^k \leq A^{km_0} B_{m,nk}^{1/m}$$

$$\eta_n^{1/n} \leq A^{m_0/n} \eta_n k^{1/nk}$$

By letting k go to infinity, we obtain

$$\eta_n^{1/n} \leq A^{m_0/n} \eta$$

□

6. Applications of the Main Theorem: entropy of 2-dimensional SFT. In this section, we provide an example of computing entropies of some shifts of finite type. Here let the alphabet set $\mathcal{A} = \{0, 1\}$

Example 6-1. $X_{\mathcal{F}}$ where $\mathcal{F} = \{ {}^{1\,1}_{1\,1} \}$

We first observe that $X_{\mathcal{F}}$ is 1-step. Moreover, by symmetry of the block ${}^{1\,1}_{1\,1}$, we have that $B_{m,n} = B_{n,m}$ for all $m, n \in \mathbb{Z}^+$ and the transfer matrix T_n is symmetric for all $n \in \mathbb{Z}^+$. Therefore we can apply the tecnique by Calkin and Wilf [**2**]. We find the transfer matrices:

$$T_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
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1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

And we also find the circular transfer matrix

$$C_4 = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&1&0&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&0&1&0&1&0&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&1&0&1&0&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0\\
1&1&1&0&1&1&0&0&1&0&1&0&0&0&0&0
\end{pmatrix}$$

We find that $\lambda_2 = 7.27938$, $\lambda_4 = 26.692$ and $\xi_4 = 13.4759$. Hence, the hard entropy is approximately computed

$$1.91489 = \left(\frac{\lambda_4}{\lambda_2}\right)^{1/2} \leq \eta \leq \xi_4^{1/4} = 1.91597$$

Example 6-2. $X_{\mathcal{F}}$ where $\mathcal{F} = \{ \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \}$

Similar to the first example, we find transfer matrices and circular transfer matrices

$$T_2 = \begin{pmatrix} 0&0&1&1&0&1&1&1\\ 0&0&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&0\\ 0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&0&0\\ 1&1&1&0&1&1&0&0 \end{pmatrix}$$

$$T_4 = \left(\begin{smallmatrix}
0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0\\
0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0\\
0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1\\
0&0&0&0&1&1&1&1&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0&0&1&1&0&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0\\
0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0\\
0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0
\end{smallmatrix}\right)$$

$$C_4 = \begin{pmatrix}
0&0&0&0&0&1&0&1&0&0&1&1&0&1&1&1\\
0&0&0&0&1&1&1&1&0&0&1&1&1&1&1&1\\
0&0&0&0&0&1&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0\\
0&1&0&1&0&1&0&1&0&1&1&1&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&1&0&0&1&1&1&1&1&1&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&1&1&0&0\\
0&0&1&1&0&1&1&1&0&0&1&1&0&1&1&1\\
0&0&1&1&1&1&1&1&0&0&1&0&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&0&1&0&1&0&1&0&1&0\\
0&1&1&1&0&1&1&1&0&1&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&0&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&1&1&1&1&0&0&0&0\\
1&1&1&0&1&1&0&0&1&0&1&0&0&0&0&0
\end{pmatrix}$$

So we evaluate $\lambda_2 = 6.41883$, $\lambda_4 = 20.8024$, and $\xi_4 = 10.6299$. Thus

$$1.800233 = \left(\frac{\lambda_4}{\lambda_2}\right)^{1/2} \le \eta \le \xi_4^{1/4} = 1.805645$$

Accordingly, this subshift has noticeably less entropy than the previous one as this subshift forbids more patterns.

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