1) Let $G \stackrel{\text{def}}{=} C_4 \times C_8$. [As usual, C_n denotes the cyclic group of order n.] Let x and y denote the generators of C_4 and C_8 respectively, i.e., $C_4 = \langle x \rangle$ and $C_8 = \langle y \rangle$, and let $H \stackrel{\text{def}}{=} \langle (x, y^7) \rangle$.

(a) Give the elements of H explicitly.

Solution.

$$\begin{aligned} H &= \left\langle (x, y^7) \right\rangle = \{(x, y^7)^k \ : \ k \in \mathbb{Z}\} \\ &= \{(1,1), (x, y^7), (x^2, y^6), (x^3, y^5), (1, y^4), (x, y^3), (x^2, y^2), (x^3, y)\}. \end{aligned}$$

□

(b) Describe G/H as a set. [In other words, give its elements.]

Solution. We know that $|G| = 4 \cdot 8 = 32$ and $|H| = 8$. Thus, $|G/H| = |G| \,/\, |H| = 4$. [This makes our lives easier, since we now have only to find three cosets besides H itself.] Since $(x, 1)$ is not in H, we have that $(x, 1)H \neq H$. We also have $(x^2, 1) \notin H, (x, 1)H$, so it gives another coset. Finally, since $(x^3, 1) \notin H, (x, 1)H, (x^2, 1)H$, we have that

$$G/H = \{H, (x, 1)H, (x^2, 1)H, (x^3, 1)H\}.$$

□

(c) To what group is G/H isomorphic? [Give a precise description, like S_3, Q_8, C_7, $C_2 \times C_2, \mathbb{Z}$, etc.]

Solution. We have that

$$G/H = \langle (x, 1)H \rangle \,,$$

and hence $G/H \cong C_4$. □

2) Let $G = (0, \infty) \times \mathbb{R}$ act on $S \stackrel{\text{def}}{=} \mathbb{R}^2$ by: given $(r, t) \in G$ and $(x, y) \in S$,

$$f_{(r,t)}(x, y) \stackrel{\text{def}}{=} (rx, y + t).$$

(a) Prove that this indeed defines a group action.

Solution.

(i) The identity of $(0, \infty) \times \mathbb{R}$ is $(1, 0)$. Then:

$$f_{(1,0)}(x, y) = (1 \cdot x, y + 0) = (x, y).$$

Thus, $f_{(1,0)}$ is the identity function.

(ii) Given $(r_1, t_1), (r_2, t_2) \in (0, \infty) \times \mathbb{R}$, we have

$$\begin{aligned} f_{(r_1,t_1)} \circ f_{(r_2,t_2)}(x, y) &= f_{(r_1,t_1)}(r_2 x, y + t_2) \\ &= (r_1 r_2 x, y + t_1 + t_2) \\ &= f_{(r_1 r_2, t_1 + t_2)}(x, y) \\ &= f_{(r_1,t_1)(r_2,t_2)}(x, y). \end{aligned}$$

□

(b) Describe the orbits of $(-\sqrt{2}, \pi)$ and $(0, 1)$.

Solution. We have:

$$\begin{aligned} O_{(-\sqrt{2},\pi)} &= \{f_{(r,t)}(-\sqrt{2}, \pi) \ : \ (r, s) \in (0, \infty) \times \mathbb{R}\} \\ &= \{(-r\sqrt{2}, \pi + t) \ : \ (r, s) \in (0, \infty) \times \mathbb{R}\} \\ &= \{(x, y) \ : \ x < 0\}. \end{aligned}$$

Hence, this orbit is the half plane on the left of the y-axis.

Also,

$$\begin{aligned} O_{(0,1)} &= \{f_{(r,t)}(0, 1) \ : \ (r, s) \in (0, \infty) \times \mathbb{R}\} \\ &= \{(0, 1 + t) \ : \ (r, s) \in (0, \infty) \times \mathbb{R}\} \\ &= \{(0, y) \ : \ y \in \mathbb{R}\}. \end{aligned}$$

Hence, this orbit is the y-axis. □

[Continues on next page!]

(c) Describe the stabilizers of $(-\sqrt{2}, \pi)$ and $(0, 1)$.

Solution. We have:

$$\begin{aligned} G_{(-\sqrt{2},\pi)} &= \{(r,t) \in G \ : \ f_{(r,t)}(-\sqrt{2}, \pi) = (-\sqrt{2}, \pi)\} \\ &= \{(r,t) \in G \ : \ (-r\sqrt{2}, \pi + t) = (-\sqrt{2}, \pi)\} \\ &= \{(1, 0)\}. \end{aligned}$$

Also,

$$\begin{aligned} G_{(0,1)} &= \{(r,t) \in G \ : \ f_{(r,t)}(0,1) = (0,1)\} \\ &= \{(r,t) \in G \ : \ (0, 1+t) = (0,1)\} \\ &= (0, \infty) \times \{0\}. \end{aligned}$$

□

3) Let G be a group with *normal* subgroups of orders 3 and 5. Prove that G has an *element* of order 15.
[If you don't think you can do this, you can try to do it with the assumption that G is Abelian. It's easier, but you will only get half of the credit.]

Solution. Let H be the subgroup of order 3 and K be the subgroup of order 5. Since $H \cap K$ is a subgroup of both H and K, its order dividers both orders, i.e., it divides both 3 and 5. Hence, $|H \cap K| = 1$, i.e., $H \cap K = \{1_G\}$.

For G Abelian: Since their orders are prime, they are both cyclic generated by any non-identity element. Let x and y be their respective generators.

We claim that xy has order 15: since G is Abelian, we have that $(xy)^k = x^k y^k$. Then $(xy)^{15} = x^{15} y^{15} = 1_G$. So, the order of xy divides 15. But $(xy)^3 = x^3 y^3 = y^3 \neq 1_G$ and $(xy)^5 = x^5 y^5 = x^2 \neq 1_G$. Hence the order of xy is indeed 15.

For G not Abelian: Now, let us not assume that G is Abelian, but that $H, K \triangleleft G$. By Proposition 2.8.6 from Artin's text, we have that $HK \cong H \times K$. [Note that we don't necessarily have that $HK = G$!!] But then, since $H \cong C_3$ and $K \cong C_5$ and $\gcd(3, 5) = 1$, we have that $H \times K \cong C_{15}$ and hence it has an element of order 15. Therefore, so does HK [and hence, since $HK \subseteq G$, so does G].

[In fact, if you look at the proof given in Proposition 2.8.6, you see that if $H, K \triangleleft G$ with $H \cap K = \{1_G\}$, then for all $h \in H$ and $k \in K$, we have $hk = kh$. (*Note that this is* not *the same as* $HK = KH$*!!!!*) But then, you can also copy the proof for Abelian groups, since the generators will commute with each other!]

□

4) Let $G \stackrel{\text{def}}{=} \mathbb{Z} \times \mathbb{Z}$ and

$$H \stackrel{\text{def}}{=} \{(n, -n) \ : \ n \in \mathbb{Z}\}.$$

Prove that $H \triangleleft G$ and $G/H \cong \mathbb{Z}$.

Solution. Let $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n, m) = n + m$.

(i) *ϕ is a homomorphism:* Let $(n_1, m_1), (n_2, m_2) \in \mathbb{Z} \times \mathbb{Z}$. Then,

$$\begin{aligned} \phi((n_1, m_1) + (n_2, m_2)) &= \phi(n_1 + n_2, m_1 + m_2) \\ &= n_1 + n_2 + m_1 + m_2 \\ &= (n_1 + m_1) + (n_2 + m_2) \\ &= \phi(n_1, m_1) + \phi(n_2, m_2). \end{aligned}$$

(ii) $\ker \phi = H$: $\phi(n, m) = 0$ iff $n + m = 0$ iff $m = -n$ iff $(n, m) \in H$. This gives us also that $H \triangleleft G$.

(iii) *ϕ is onto:* given $n \in \mathbb{Z}$, we have $\phi(n, 0) = n$.

Therefore, by the *First Isomorphism Theorem*, we have that $G/H \cong \mathbb{Z}$.

□