Extra Credit Problem

(Due in class on Monday 10/30.)

Math 455

Problem from this years "UT Math Contest" (Fermat II) for high school students.

Problem: Let $a, b, c \in \{1, 2, \ldots, 2005\}$ and

$$f(X) \stackrel{\text{def}}{=} aX^{101} + bX^{100} + c.$$

Prove that if $f(2006)$ is prime, then $f(X)$ has no integral root, i.e., there is no $n \in \mathbb{Z}$ such that

$$a\, n^{101} + b\, n^{100} + c = 0.$$

Proof. Suppose that $n \in \mathbb{Z}$ is a root. Then

$$c = -a\, n^{101} - b\, n^{100} = -n^{100}(an + b).$$

Since, $c > 0$, we must have $an + b < 0$, and since $a, b > 0$, we must have $n < 0$.

If $n \leq -2$, then

$$c \geq -(-2)^{100}(-2a + b) = 2^{100}(b - 2a) \geq 2^{100} > 2005.$$

[Remember that $-2a + b < 0$.] But this cannot happen [since $c \leq 2005$].

So, [since $n < 0$ and $n \geq -2$] we must have $n = -1$, and thus $c = a - b$. Then,

$$f(2006) = a\, 2006^{101} + b\, 2006^{100} + (a - b) = a\,(2006^{101} + 1) + b\,(2006^{100} - 1)$$

is a prime. On the other hand $2006 \equiv -1 \pmod 3$ [since $2007 = 2006 + 1$ is divisible by 3.] So,

$$\begin{aligned} f(2006) &\equiv a\,(2006^{101} + 1) + b\,(2006^{100} - 1) \\ &\equiv a\,((-1)^{101} + 1) + b\,((-1)^{100} - 1) \\ &\equiv a(-1 + 1) + b(1 - 1) \\ &\equiv 0 \pmod 3. \end{aligned}$$

Thus, 3 divides the prime $f(2006)$, and so this prime should be 3. But,

$$f(2006) = a\,2006^{101} + b\,2006^{100} + c > 2006^{101} + 2006^{100} + 1$$

[since $a, b, c > 0$], and so it cannot be 3.

Therfore, if $f(2006)$ is prime, then $f(X)$ has no integral root.

$\square$