1) Suppose that $|G| = 2p$, where p is a prime different from 2. Prove that either $G \cong C_{2p}$ or $G \cong D_{2p}$.

Proof. By the First Sylow Theorem, [since 2 and p are both primes and $p \neq 2$] there are subgroups H and K such that $|H| = p$ and $|K| = 2$. Hence, since they have prime orders, $H \cong C_p$ and $K \cong C_2$. Let $H = \langle x \rangle$ and $K = \langle y \rangle$.

Since $[G : H] = 2$, we have that $H \triangleleft G$. [We could also obtain that from Third Sylow Theorem.] We also have that $H \cap K = \{1\}$ [since their orders are relatively prime], and, by Proposition 2.8.6(a), since $|H| \cdot |K| = |G|$, we have $H \cdot K = G$. Therefore,

$$G = \{1, x, x^2, \ldots, x^{p-1}, y, xy, x^2y, \ldots, x^{p-1}y\}.$$

If $K \triangleleft G$, then we have, by Proposition 2.8.6(c), that $G \cong H \times K \cong C_p \times C_2 \cong C_{2p}$. [In the last equality, we used the fact that $p \neq 2$.]

Suppose then that K is not normal. By the Second Sylow Theorem, we have that there is more than one Sylow 2-subgroup, while there is only one Sylow p-subgroup [namely, H]. By the Third Sylow Theorem, s_2 [i.e., the number of Sylow 2-subgroups] divides p, so it is either 1 or p. Since it is not 1 [as we've seen above], it must be p. So, we have p elements of orders 2. Since all p elements of H do not have order 2 [they have order p or 1], all other elements must have order 2. So, $y, xy, \ldots, x^{p-1}y$ all have order two. So, xy has order two, and:

$$(xy)^2 = xyxy = 1 \quad \Rightarrow \quad yx = x^{-1}y^{-1} = x^{-1}y$$

[since y also has order two]. Thus, $G = \langle x, y \rangle$, x has order p, y has order 2, and $yx = x^{-1}y$. Therefore $G \cong D_{2p}$.

□

2) Let $H \triangleleft G$, $\bar{K} < G/H$, and

$$K \stackrel{\text{def}}{=} \{x \in G \ : \ x \in gH \text{ for some } gH \in \bar{K}\}$$

[i.e., K is the union of all cosets in $\bar{K}$].

(a) Prove that K is a subgroup of G containing H.

Solution. Let $x, y \in K$. So, [by defn. of K] there are $g_1H, g_2H \in \bar{K}$ such that $x \in g_1H$ and $y \in g_2H$. Thus, $y^{-1} \in Hg_2^{-1} = g_2^{-1}H = (g_2H)^{-1}$ [since $H \triangleleft G$ and $\bar{K} < G/H$]. Therefore $xy^{-1} \in (g_1H)(g_2H)^{-1}$. Since $\bar{K} < G/H$, we have that $(g_1H)(g_2H)^{-1} = (g_1g_2^{-1})H \in \bar{K}$. Hence, $xy^{-1} \in K$. By the one-step method, $K < G$.

Now, since $1 \cdot H = H \in \bar{K}$, all its elements are in K.

□

(b) Prove that $\bar{K} = \{kH \ : \ k \in K\}$.

Solution. Let $gH \in \bar{K}$. Then $g \cdot 1 = g \in K$. Therefore, $gH \in \{kH \ : \ k \in K\}$, and $\bar{K} \subseteq \{kH \ : \ k \in K\}$.

Let $k \in K$. Then $k \in gH$ for some $gH \in \bar{K}$. So, $kH = gH$ [since the cosets are disjoint]. Hence, $kH \in \bar{K}$, and $\{kH \ : \ k \in K\} \subseteq \bar{K}$.

Thus, $\bar{K} = \{kH \ : \ k \in K\}$.

□

3) Let $M, N \triangleleft G$.

(a) Prove that $(NM) < G$, $M \triangleleft (NM)$, and $(N \cap M) \triangleleft N$.

Solution. Since, $M, N \triangleleft G$, by Proposition 2.8.6(b), $NM < G$.

Let $m \in M$ and $g \in NM$. Since $NM \subseteq G$ and $M \triangleleft G$, $gmg^{-1} \in M$, and so $M \triangleleft NM$.

We will prove that $(N \cap M) \triangleleft N$ in (b) below. [Or, you can just quote Proposition 2.7.1.]

□

(b) Prove that $N/(N \cap M) \cong (NM)/M$.

Solution. Let $\phi : N \to (NM)/M$ defined by $\phi(n) = nM$. [Note that since $N \subseteq NM$, we have $nM \in (NM)/M$.]

We have $\phi(n_1 n_2) = (n_1 n_2)M = (n_1 M)(n_2 M)$, and hence ϕ is a homomorphism.

Given $nmM \in (NM)/M$, we have that $nmM = nM$, since $nmm^{-1} = n \in nmM$ [and cosets are disjoint]. So, $\phi(n) = nM = nmM$, and ϕ is onto.

We have that $\phi(n) = M$ iff $nM = M$ iff $n \in M$. Since we also have that $n \in N$, we obtain $\ker \phi = N \cap M$. [In particular, this proves that $(N \cap M) \triangleleft N$ for part (a).]

By the *First Isomorphism Theorem*, $N/(N \cap M) \cong (NM)/M$.

□

4) Let G be an *Abelian group*, $H < G$ and $\phi : G \to H$ be a homomorphism such that $\phi(h) = h$ for all $h \in H$. Prove that $G \cong H \times \ker \phi$. [**Hint:** Remember that $\phi(g) = \phi(g')$ iff $g^{-1}g' \in \ker \phi$.]

Solution. Yet again, we use Proposition 2.8.6.

[$H, \ker \phi \lhd G$:] Since G is Abelian, both H and $\ker \phi$ are normal subgroups of G.

[$H \cap \ker \phi = \{1\}$:] Let $g \in H \cap \ker \phi$. In particular $g \in H$, and so $\phi(g) = g$. On the other hand, also $g \in \ker \phi$, and so $\phi(g) = 1$. Thus, $g = \phi(g) = 1$, and $H \cap \ker \phi = \{1\}$ [since we proved that an arbitrary element of $H \cap \ker \phi$ has to be equal to 1].

[$H \cdot \ker \phi = G$:] Let $g \in G$. Then $\phi(g) \in H$. So, denote $h \stackrel{\text{def}}{=} \phi(g)$. Then, since $h \in H$, we have that $\phi(h) = h = \phi(g)$. By the hint, $h^{-1}g \in \ker \phi$. But then, $g = h \cdot (h^{-1}g) \in H \cdot \ker \phi$. Since g was arbitrary, we have $H \cdot \ker \phi = G$.

By Proposition 2.8.6(c), $G \cong H \times \ker \phi$.

$\square$

5) Let R be a [not necessarily commutative] ring in which $a^2 = a$ for *all* $a \in R$.

(a) Prove that for all $a \in R$, we have $a = -a$.

Solution. We have $-a = (-a)^2 = (-a)(-a) = a^2 = a$. [Remember that it was proved in class that $(-x)(-y) = xy$.]

$\square$

(b) Prove that R is commutative. [**Hint:** Expand $(a+b)^2$ in the ring.]

Solution. We have

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a + ab + ba + b.\end{aligned}$$

On the other hand, $(a+b)^2 = (a+b)$. So,

$$\begin{aligned}a + ab + ba + b = a + b \quad &\Rightarrow \quad ab + ba = 0\\ &\Rightarrow \quad ab = -ba\\ &\Rightarrow \quad ab = ba.\end{aligned}$$

[where the last statement comes from part (a)].

$\square$