Quadratic Equation

Math 455 – Fall 2006

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For given values of a, b and c, we want to find an x such that:

$$ax^2 + bx + c = 0. \tag{1}$$

Note that if we had an equation of the form

$$(px + q)^2 + r = 0, \tag{2}$$

then it would be easy to find a solution:

$$\begin{aligned}
(px + q)^2 + r = 0 &\quad \Rightarrow \\
(px + q)^2 = -r &\quad \Rightarrow \\
px + q = \pm\sqrt{-r} &\quad \Rightarrow \\
px = -q \pm \sqrt{-r} &\quad \Rightarrow \\
x = \frac{-q \pm \sqrt{-r}}{p} &
\end{aligned}$$

So, we want to go from equation (1), to something similar to equation (2). In order to do so, we *complete the square*. Remember that for all X and Y,

$$(X + Y)^2 = X^2 + 2XY + Y^2. \tag{3}$$

Hence:

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = ax^2 + bx + \left(\frac{b^2}{4a}\right),$$

or,

$$ax^2 + bx = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \left(\frac{b^2}{4a}\right). \tag{4}$$

Therefore:

$$ax^2 + bx + c = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + \left[c - \left(\frac{b^2}{4a}\right)\right]. \tag{5}$$

So, we are now in the same situation as equation (2) with

$$p = \sqrt{a}, \qquad q = \frac{b}{2\sqrt{a}}, \qquad r = c - \left(\frac{b^2}{4a}\right).$$

So,

$$\begin{aligned}
x &= \frac{-q \pm \sqrt{-r}}{p} \\
&= \frac{-b/(2\sqrt{a}) \pm \sqrt{b^2/(4a) - c}}{\sqrt{a}} \\
&= \frac{-b \pm 2\sqrt{a} \cdot \sqrt{b^2/(4a) - c}}{2a} \quad \text{[multiply top and bottom by } 2\sqrt{a}.\text{]} \\
&= \frac{-b \pm \sqrt{4a} \cdot \sqrt{b^2/(4a) - c}}{2a} \\
&= \frac{-b \pm \sqrt{4a \cdot (b^2/(4a) - c)}}{2a} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$