1) [10 points] Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4, 2x_1 - 4x_2 + 3x_3 + x_4, x_3 + x_4, x_1 - 2x_2 + x_3 + 2x_4).$$

Find all vectors $\mathbf{x} \in \mathbb{R}^4$ [if any] such that $T(\mathbf{x}) = (2, 0, 0, 2)$ and all vectors [if any] such that $T(\mathbf{x}) = (1, 1, 1, 1)$.

Solution. We have to simultaneously solve the systems:

$$\left[\begin{array}{rrrr|r|r} 1 & -2 & 0 & 1 & 2 & 1 \\ 2 & -4 & 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 2 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{rrrr|r|r} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Hence, there is no $\mathbf{x}$ such that $T(\mathbf{x}) = (1, 1, 1, 1)$, and all $\mathbf{x}$ such that $T(\mathbf{x}) = (2, 0, 0, 2)$ are of the form $(1 + 2t, t, -1, 1)$ for $t \in \mathbb{R}$.

□

2) [10 points] Fill in the blanks in the table below. [No need to justify this one.] Here T_A, as usual, represents the linear transformation associated to the matrix A.

size of A	3×3	3×3	3×3	5×9	9×5	3×1	6×5
rank of A	**2**	3	1	1	5	0	5
rank of A^{T}	2	3	1	1	5	0	5
dim. of row spc.	2	3	**1**	1	5	0	5
dim. of col spc.	2	3	1	1	5	0	5
nullity of A	1	0	2	8	0	1	**0**
nullity of A^{T}	1	0	2	**4**	4	3	1
T_A 1-to-1 (y/n)	N	Y	N	N	**Y**	**N**	Y
T_A onto (y/n)	N	**Y**	N	N	N	N	N

3) Let $S = \{(1,0,1,2,1),(0,1,1,-1,2),(-1,2,1,-4,3),(2,1,2,-1,1),(0,0,1,4,3)\}$ and let $V = \operatorname{span} S$ [the subspace of $\mathbb{R}^5$ spanned by the set S]. Given that

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 2 \\ -1 & 2 & 1 & -4 & 3 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -5 & -1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & -1 & -4 & -1 & 4 \\ 1 & 2 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the following.

(a) [5 points] Find a basis of V made of vectors in S.

Solution. We use the reduced echelon form of the matrix that has the vectors as columns. The columns with leading ones are the first, second, and fourth, so first, second, and fourth vectors of S form a basis of $\operatorname{span}(S)$. □

(b) [5 points] If B is the basis you've found in part (a), express the vectors in S that are not in B as a linear combination of vectors in B.

Solution. We again use the reduced echelon form of the matrix that has the vectors as columns. If the columns of this matrix are $\mathbf{c}_1'$ through $\mathbf{c}_5'$, we can easily see that

$$\mathbf{c}_3' = -1\mathbf{c}_1' + 2\mathbf{c}_2' \qquad \text{and} \qquad \mathbf{c}_5' = 2\mathbf{c}_1' + \mathbf{c}_2' + (-1)\mathbf{c}_4'.$$

If $\mathbf{c}_1$ through $\mathbf{c}_5$ now denote the vectors in S [which appear as columns of the second matrix], and so $B = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$, we have:

$$\mathbf{c}_3 = -1\mathbf{c}_1 + 2\mathbf{c}_3 \qquad \text{and} \qquad \mathbf{c}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + (-1)\mathbf{c}_4.$$

□

(c) [5 points] Find a basis for the orthogonal complement $V^\perp$ of V.

Solution. We have that the orthogonal complement of V is the nullspace of the matrix that has the vectors of S as rows. Using the given reduced echelon form of this matrix, we get that the nullspace is given by:

$$(2s + 2t, 5s + t, -4s - 3t, s, t) = s(2, 5, -4, 1, 0) + t(2, 1, -3, 0, 1),$$

for $s, t \in \mathbb{R}$. Hence, the basis of $V^\perp$ is $\{(2, 5, -4, 1, 0), (2, 1, -3, 0, 1)\}$.

□

(d) [5 points] Find a basis of $\mathbb{R}^5$ containing a basis of V you found in part (b).

Solution. We just need to add the standard basis elements of $\mathbb{R}^5$ that have leading ones in the places where we are missing leading ones in the reduced echelon form of the matrix that has the elements of S are rows [i.e., the first matrix]. Hence, we have the following basis of $\mathbb{R}^5$:

$$\{(1, 0, 1, 2, 1), (0, 1, 1, -1, 2), (2, 1, 2, -1, 1), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}.$$

□

4) Let P_2 be the vector space of polynomials of degree less than or equal to 2. For $\mathbf{p}$ and $\mathbf{q}$ in P_2, define the inner product by:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1/2)\mathbf{q}(1/2) + \mathbf{p}(1)\mathbf{q}(1).$$

For example, $\langle x, x^2+1 \rangle = 0 \cdot 1 + (1/2) \cdot (5/4) + 1 \cdot 2 = 21/8$.

(a) [5 points] Is $B' = \{2x^2 - 3x + 1, 2x^2 - x, -4x^2 + 4x\}$ an *orthonormal* basis of P_2?

Solution. Let $\mathbf{p}_1 = 2x^2 - 3x + 1$, $\mathbf{p}_2 = 2x^2 - x$, and $\mathbf{p}_3 = -4x^2 + 4x$. Then,

$$\begin{aligned} &\mathbf{p}_1(0) = 1,\ \mathbf{p}_1(1/2) = \mathbf{p}_1(1) = 0, \\ &\mathbf{p}_2(0) = \mathbf{p}_2(1/2) = 0,\ \mathbf{p}_2(1) = 1, \\ &\mathbf{p}_3(0) = \mathbf{p}_3(1) = 0,\ \mathbf{p}_3(1/2) = 1. \end{aligned}$$

Hence,

$$\begin{array}{lll} \langle \mathbf{p}_1, \mathbf{p}_1 \rangle = 1 & \langle \mathbf{p}_2, \mathbf{p}_2 \rangle = 1 & \langle \mathbf{p}_3, \mathbf{p}_3 \rangle = 1 \\ \langle \mathbf{p}_1, \mathbf{p}_2 \rangle = 0 & \langle \mathbf{p}_1, \mathbf{p}_3 \rangle = 0 & \langle \mathbf{p}_2, \mathbf{p}_3 \rangle = 0. \end{array}$$

Hence, it is an orthonormal basis.

□

(b) [5 points] Find $(1)_{B'}$, $(x)_{B'}$, and $(x^2)_{B'}$.

Solution. Since the basis is orthonormal, we have

$$(\mathbf{p})_{B'} = (\langle \mathbf{p}, \mathbf{p}_1 \rangle, \langle \mathbf{p}, \mathbf{p}_2 \rangle, \langle \mathbf{p}, \mathbf{p}_3 \rangle).$$

Thus,

$$\begin{aligned} (1)_{B'} &= (1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0, 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 1, 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0,) \\ &= (1, 1, 1), \\ (x)_{B'} &= (0 \cdot 1 + 1/2 \cdot 0 + 1 \cdot 0, 0 \cdot 0 + 1/2 \cdot 0 + 1 \cdot 1, 0 \cdot 0 + 1/2 \cdot 1 + 1 \cdot 0,) \\ &= (0, 1, 1/2), \\ (x^2)_{B'} &= (0 \cdot 1 + 1/4 \cdot 0 + 1 \cdot 0, 0 \cdot 0 + 1/4 \cdot 0 + 1 \cdot 1, 0 \cdot 0 + 1/4 \cdot 1 + 1 \cdot 0,) \\ &= (0, 1, 1/4). \end{aligned}$$

□

(c) [5 points] If $B = \{1, x, x^2\}$ is the standard basis of P_2, find the transition matrix from B to B'.

Solution. We just need to put the coordinates we found in the previous item as columns:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1/2 & 1/4 \end{bmatrix}$$

□

(d) [5 points] Suppose $(\mathbf{p})_{B'} = (1, 0, -2)$ and $(\mathbf{q})_{B'} = (1, 1, 1)$ and let θ be the angle between them [with respect to the given inner product]. Compute, $\langle \mathbf{p}, \mathbf{q} \rangle$, $\|\mathbf{p}\|$, and $\cos\theta$.

Solution. Since the basis is orthonormal, we have:

$$\langle \mathbf{p}, \mathbf{q} \rangle = 1 \cdot 1 + 0 \cdot 1 + (-2) \cdot 1 = -1,$$

$$\|\mathbf{p}\| = \sqrt{1^2 + 10^2 + (-2)^2} = \sqrt{5},$$

$$\cos\theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\|\mathbf{p}\| \, \|\mathbf{q}\|} = \frac{-1}{\sqrt{5}\sqrt{3}} = -\frac{\sqrt{15}}{15}.$$

□

5) [10 points] Let

$$A = \begin{bmatrix} 9 & 8 & 8 \\ -5 & -7 & -11 \\ 5 & 8 & 12 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Then, we have that $P^{-1}AP = D$. [You don't have to check it! Just take my word for it.] Find a matrix B such that $B^2 = A$. [So, in some sense, $B = A^{1/2}$, where A is diagonalizable.]

Solution. Let

$$D' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then, $(D')^2 = D$. Hence,

$$A = PDP^{-1} = P(D')^2P^{-1} = (PD'P^{-1})(PD'P^{-1}) = (PD'P^{-1}).$$

So, we can take $B = (PD'P^{-1})$. Explicitly:

$$\begin{aligned} B &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 3 \\ 2 & 2 & -3 \\ -2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & 2 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \end{aligned}$$

□

6) Let

$$A = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}.$$

(a) [5 points] Give the characteristic equation of A and show that the eigenvalues are 0 and -6.

Solution. The characteristic polynomial is:

$$\begin{aligned} \det(xI - A) &= \begin{vmatrix} (x+4) & -2 & -2 \\ -2 & (x+4) & -2 \\ -2 & -2 & (x+4) \end{vmatrix} \\ &= (x+4)^3 - 8 - 8 - (4(x+4) + 4(x+4) + 4(x+4)) \\ &= (x^3 + 12x^2 + 48x + 64) - 16 - 12x - 48 \\ &= x^3 + 12x^2 + 36x \\ &= x(x+6)^2 \end{aligned}$$

So, the characteristic equation is $x(x+6)^2 = 0$ and the eigenvalues are 0 and -6.

□

(b) [5 points] Find the eigenspaces for each eigenvalue. [Since I've given you the eigenvalues, i.e., 0 and -6, you can do this part even if you didn't do part (a).]

Solution. The eigenspace of associated to 0 is given by the nullspace of $0 \cdot I - A = -A$. Since,

$$-A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

we have that the eigenspace is $\operatorname{span}\{(1,1,1)\}$.

The eigenspace of associated to -6 is given by the nullspace of

$$(-6)I - A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we have that the eigenspace is $\operatorname{span}\{(-1,1,0),(-1,0,1)\}$.

□

(c) [5 points] Find an *orthogonal* matrix P such that $P^{-1}AP$ is diagonal.

Solution. First, we need to make the basis of the eigenspaces orthonormal. For the eigenspace of 0, we get $\{(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)\}$.

For the eigenspace of -6 we have that the component of $(-1, 0, 1)$ orthogonal to $(-1, 1, 0)$ is

$$\begin{aligned}(-1,0,1) - \text{proj}_{(-1,1,0)}(-1,0,1) &= (-1,0,1) - \frac{(-1)\cdot(-1) + 0\cdot 1 + 1\cdot 0}{(-1)^2 + 1^2 + 0^2}(-1,1,0) \\ &= (-1,0,1) - \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\ &= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)\end{aligned}$$

Hence $\{(-1, 1, 0), (-1/2, -1/2, 1)\}$ is an orthogonal basis of this eigenspace, and the orthonormal basis is [divide the vectors by their lenghts]

$$\left\{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)\right\}.$$

Now, we just need to take P as the change of basis matrix from this basis of eigenvalues to the standard basis. So, we just put the new basis as columns of P:

$$P = \begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

□

(d) [5 points] Give the diagonal matrix $P^{-1}AP$.

Solution. Note that we can do this without doing the previous one. Since we know that the characteristic equation is $x(x + 6)^2 = 0$ and A is diagonalizable, we have that

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

□