1) [10 points] Let $X = \{1, 2, 3\}$. Give an example of a relation $\mathcal{R}$ [not necessarily an equivalence relation!] on X such that $1\mathcal{R}2$, but 1 is *not* related to 3 by $\mathcal{R}$.

Solution. There are may possibilities. The simplest one is to take $\mathcal{R} = \{(1, 2)\}$. Then, $1\mathcal{R}2$ [as $(1, 2) \in \mathcal{R}$], but 1 is not related to 3 [as $(1, 3) \notin \mathcal{R}$]. $\square$

2) [10 points] Give the definition of *domain* of a relation $\mathcal{R}$ on $X \times Y$.

Solution. The domain of $\mathcal{R}$ is defined as the set:

$$\{x \in X \ : \ \exists y \in Y \text{ such that } x\mathcal{R}y\}.$$

$\square$

3) [10 points] Let $X = \{1, 2, 3, 4, 5\}$. Is $\mathcal{P} = \{\{1, 2\}, \{3, 4\}, \{2, 5\}\}$ a partition of X? How about $\mathcal{Q} = \{\{1, 2\}, \{3\}, \{4\}\}$? Justify your answers!

Solution. $\mathcal{P}$ is not a partition, as $\{1, 2\}, \{2, 5\} \in \mathcal{P}$, with $\{1, 2\} \cap \{2, 5\} = \{2\} \neq \varnothing$, but $\{1, 2\} \neq \{2, 5\}$. [Distinct elements of a partition have to be disjoint!]

$\mathcal{Q}$ is not a partition either, as 5 is in X, but not in any element of $\mathcal{Q}$. [A partition must cover all elements of the set.]

$\square$

4) [15 points] Let x be a real number. Prove that if x is irrational, then $1/x$ is also irrational.

Proof. We prove the contrapositive. [I.e., of $1/x \in \mathbb{Q}$, then $x \in \mathbb{Q}$.]

Suppose that $1/x \in \mathbb{Q}$. [We need to prove that $x \in \mathbb{Q}$.] Then, clearly $1/x \neq 0$ [as $a/b = 0$ if, and only if $a = 0$], and since $1 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under division by *nonzero elements*, we have that $\dfrac{1}{1/x} = x \in \mathbb{Q}$. $\square$

5) [10 points] Prove or disprove: Let A, B and C be sets. Then, $[(A\setminus B)\setminus C]\cup(B\setminus C) = A\setminus C$.

Solution. The statement is FALSE. [Draw the Venn diagrams.] Here is a counter-example: take $A = C = \varnothing$ and $B = \{1\}$. Then, $(A \setminus B) \setminus C = \varnothing$ and $B \setminus C = \{1\}$. Thus, $[(A \setminus B) \setminus C] \cup (B \setminus C) = \{1\}$. On the other hand, $A \setminus C = \varnothing$. Thus, in this case, $[(A \setminus B) \setminus C] \cup (B \setminus C) \neq A \setminus C$.

$\square$

6) [15 points] Prove or disprove: Let A, B and C be sets. Then, $(A \setminus B) \cap C = (C \setminus B) \cap A$.

Proof. The statement is TRUE.

[“$\subseteq$”:] Let $x \in (A \setminus B) \cap C$. Then, $x \in A \setminus B$ and $x \in C$. Thus, $x \in A$, $x \notin B$, $x \in C$. Therefore, $x \in C \setminus B$ [as $x \in C$, $x \notin B$] and $x \in A$. Hence, $x \in (C \setminus B) \cap A$.

[“$\subseteq$”:] Let $x \in (C \setminus B) \cap A$. Then, $x \in C \setminus B$ and $x \in A$. Thus, $x \in C$, $x \notin B$, $x \in A$. Therefore, $x \in A \setminus B$ [as $x \in A$, $x \notin B$] and $x \in C$. Hence, $x \in (A \setminus B) \cap C$.

$\square$

7) [10 points] Give the completely simplified negation of the following statement. [Your answers should have no "nots" in them.]

> *Given $x > 0$, there exists y such that (if $x < 1$, then $y < x$) and (if $x \geq 1$, then $y \geq x$).*

[**Hint:** Negate one part at a time, as it makes it easier to get partial credit.]

Solution. We can rephrase it as:

$$\forall x > 0, \exists y \text{ such that } (x < 1 \Rightarrow y < x) \text{ and } (x \geq 1 \Rightarrow y \geq x).$$

Then

$$\begin{aligned}
&\text{NOT}[\forall x > 0, \exists y \text{ such that } (x < 1 \Rightarrow y < x) \text{ and } (x \geq 1 \Rightarrow y \geq x)] \\
&\qquad = \exists x > 0 \text{ such that NOT}[\exists y \text{ such that } (x < 1 \Rightarrow y < x) \text{ and } (x \geq 1 \Rightarrow y \geq x)] \\
&\qquad = \exists x > 0 \text{ such that } \forall y \text{ NOT}[(x < 1 \Rightarrow y < x) \text{ and } (x \geq 1 \Rightarrow y \geq x)] \\
&\qquad = \exists x > 0 \text{ such that } \forall y \text{ NOT}[(x < 1 \Rightarrow y < x)] \text{ or NOT}[(x \geq 1 \Rightarrow y \geq x)] \\
&\qquad = \exists x > 0 \text{ such that } \forall y \ (x < 1 \text{ and } y \geq x) \text{ or } (x \geq 1 \text{ and } y < x).
\end{aligned}$$

$\square$

8) Let $X = \mathbb{R} \setminus \{0\}$. Define a relation $\mathcal{R}$ by $x\mathcal{R}y$ if $x/y \in \mathbb{Q}$.

(a) [15 points] Prove that $\mathcal{R}$ is an equivalence relation.

Proof. [Reflexive:] Let $x \in X$. Then, $x \neq 0$ [this is *crucial!*] and so $x/x = 1$. Since $1 \in \mathbb{Q}$, we have that $x\mathcal{R}x$.

[Symmetric:] Suppose that $x\mathcal{R}y$. Then, by definition, $x/y \in \mathbb{Q}$. Since $x \neq 0$, we have also that $x/y \neq 0$. Thus, since $1 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under division by *nonzero elements* [and $x/y \neq 0$], we have that $\dfrac{1}{x/y} = y/x \in \mathbb{Q}$. Therefore, we have that $y\mathcal{R}x$ by definition.

[Transitive:] Suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then, by definition we have that $x/y, y/z \in \mathbb{Q}$. Since $\mathbb{Q}$ is closed under products, we have that $(x/y) \cdot (y/z) = x/z \in \mathbb{Q}$. Therefore $x\mathcal{R}z$ by definition.

□

(b) [5 points] Describe the equivalence class of 1. [Your answer should be a *simple* and well described subset of X. Just writing the definition is not enough, although it'd get you some partial credit.]

Solution. We have:

$$\begin{aligned} \bar{1} &= \{x \in X \ : \ x\mathcal{R}1\} \\ &= \{x \in X \ : \ x/1 \in \mathbb{Q}\} \\ &= \{x \in X \ : \ x \in \mathbb{Q}\}. \end{aligned}$$

So, $\bar{1} = X \cap \mathbb{Q} = \mathbb{Q} \setminus \{0\}$.

□