November 4th, 2008

Math 300

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Fall 2008

Name: ....

Student ID (last 6 digits): XXX- ....

Midterm 2

You have 75 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 7 questions and 10 printed pages (including this one and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	20	
7	20	
Total	100	

1) [10 points] Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(a) f is one-to-one, but not onto.

(b) f is onto, but not one-to-one.

(c) f is neither one-to-one nor onto.

2) [10 points] Prove that

$$\sum_{k=1}^{n} \left(k^2 - \frac{k}{3} \right) = \frac{n^2(n+1)}{3},$$

for all integers $n \geq 1$.

3) [10 points] Show that for all integers $n \geq 1$, we have that 5 divides $4^{2n-1} + 1$.

4) [15 points] Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -x, & \text{if } x \in \mathbb{Q}, \\ x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

So, for instance, $f(1/2) = -1/2$ and $f(\sqrt{2}) = \sqrt{2}$. Is f onto $\mathbb{R}$? Is it one-to-one? If both, find its inverse. [As always, justify your answers!]

5) [15 points] Prove that for all integers $n \geq 4$, we have:

(a) $2n + 1 < 2^n$

(b) $n^2 \leq 2^n$

6) [20 points] Let $f : X \to Y$ be a function, and $A, B \subseteq X$. [You cannot quote previous work on this question, as we've done all these questions already.]

(a) Prove that $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

(b) Disprove that $f(A \setminus B) \subseteq f(A) \setminus f(B)$ [in general].

[Part (c) on the next page]

(c) Prove that if f is one-to-one, then we do have $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

7) [20 points] Let $f : X \to Y$ be a function and $A \subseteq Y$. Show that if f is onto, then $f(f^{-1}(A)) = A$. Show that this is not necessarily true if f is not onto. [Again, this was done before, so you cannot just quote the result.]

Scratch: