1) Negate the following statement: *"given $x \in (0, \infty)$, we have that $x > y > 0$ for some $y \in (0, \infty)$"*. [As usual, your answer should contain no "nots".]

Solution. There exists $x \in (0, \infty)$, such that for all $y \in (0, \infty)$, $x \leq y$ or $y \leq 0$. $\square$

2) Let A, B, C be subsets of U. [You can use $U = \mathbb{R}$ if you want, but it is not necessary.] Prove of disprove: $(A \setminus C) \cup (A \cap B) = A \cap ((B \cap C)^{c})$.

Solution. The statement is false. Let $A = C = \{1\}$, $B = \varnothing$, and $U = \mathbb{R}$. Then, $(A \setminus C) \cup (A \cap B) = \varnothing \cup \varnothing = \varnothing$, while $A \cap ((B \cap C)^{c}) = \{1\} \cap (\varnothing^{c}) = \{1\} \cap \mathbb{R} = \{1\}$. Hence, they are different.

$\square$

3) Let $\sim$ be the relation on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by $(a, b) \sim (c, d)$ iff there exists $r \in \mathbb{R}$ such that $(ra, rb) = (c, d)$. Prove that $\sim$ is an equivalence relation.

Proof. [Reflexive:] $(a, b) \sim (a, b)$ as $(a, b) = (1 \cdot a, 1 \cdot b)$.

[Symmetric:] Suppose that $(a, b) \sim (c, d)$. Hence, $(c, d) = (ra, rb)$ for some $r \in \mathbb{R}$. Since $(c, d) \neq (0, 0)$, we must have that $r \neq 0$. Thus, $(a, b) = (1/r \cdot c, 1/r \cdot d)$ [and $1/r \in \mathbb{R}$]. Thus, $(c, d) \sim (a, b)$.

[Transitive:] Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then, by definition, there are $r, s \in \mathbb{R}$ such that $(c, d) = (ra, rb)$ and $(e, f) = (sc, sd)$. Thus, $(e, f) = (rsa, rsb)$. Since $rs \in \mathbb{R}$, we have that $(a, b) \sim (e, f)$.

$\square$

4) Let $f : X \to Y$ be a one-to-one function and $A, B \subseteq X$. Prove that $f(A \cap B) = f(A) \cap f(B)$.

Proof. ["$\subseteq$:"] Let $y \in f(A \cap B)$. Then, there exists $x \in A \cap B$ such that $f(x) = y$. Since $x \in A$, we have that $y = f(x)$ is in $f(A)$, and since $x \in B$ also, we have that $y = f(x) \in f(B)$. Hence, $y \in f(A) \cap f(B)$.

["$\supseteq$:"] Let $y \in f(A) \cap f(B)$. Then, $y \in f(A)$ and $y \in f(B)$. The former tells us that there exists $x \in A$ such that $y = f(x)$, while the latter tells us that there exists $x' \in B$ such that $y = f(x')$. Since f is one-to-one and $f(x) = f(x')$, we must have that $x = x'$. Thus, $x \in A \cap B$.

$\square$

5) Show that for all positive integers n, we have that $n^7 - n$ is divisible by 7.

Proof. We prove it by induction on n. For $n = 1$, we have $n^7 - n = 0$, which is divisible by 7.

Now suppose that $n^7 - n$ is divisible by 7. Then,

$$\begin{aligned}(n+1)^7 - (n+1) &= n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1 - (n-1) \\ &= [n^7 - n] + [7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n] \\ &= [n^7 - n] + 7 \cdot [n^6 + 3n^5 + 5n^4 + 5n^3 + 3n^2 + n].\end{aligned}$$

[Note that we can compute $(n+1)^7$ quickly using Pascal's Triangle!] Hence, $(n+1)^7 - (n+1)$ is, using the induction hypothesis, a sum of two terms divisible by 7, and hence is itself divisible by 7.

□

6) Prove that $n! \leq n^n$ for all positive integers n.

Proof. Before we prove the result, we need the following. If $0 < a < b$, then $a^n < b^n$ for all positive integer n. We prove it by induction: for $n = 1$, it is trivial. Now suppose that $a^n < b^n$. If we use Problem 9 below, we have that $0 < a < b$ and $0 < a^n < b^n$ implies that $a^{n+1} < b^{n+1}$.

We prove it by induction on n. For $n = 1$, we have that $1! = 1^1$.

Now assume that $n! \leq n^n$. Then,

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ &\leq (n+1) \cdot n^n && \text{[by the IH]} \\ &\leq (n+1) \cdot (n+1)^n && \text{[by the above, as } 0 < n < n+1\text{]} \\ &= (n+1)^{n+1}.\end{aligned}$$

□

7) Find a closed formula for the recursion $a_0 = 0$, $a_n = 3 \cdot a_{n-1} + 2$ for $n \geq 1$. [You don't have to show me how you came up with the formula, but you have to prove that it is correct.]

Proof. We prove that $a_n = 3^n - 1$ by induction on n. For $n = 0$, we have $a_0 = 0 = 3^0 - 1$. So, now assume that $a_n = 3^n - 1$ for some $n \geq 0$. Then, $a_{n+1} = 3a_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 3 + 2 = 3^{n+1} - 1$.

□

8) Let F be a field and $a, b \in F$. Also, let $n(x)$ denote the additive inverse of x [which I denoted by $-x$] and $q(x)$ denote the multiplicative inverse of x [which I denoted by x^{-1}]. *Using only the field axioms* show that:

(a) $n(a+b) = n(a) + n(b)$

Proof. We need to show that $(a+b) + (n(a) + n(b)) = 0$. We have:

$$\begin{aligned}
(a+b) + (n(a)+n(b)) &= (a+b) + (n(b)+n(a)) && \text{[by comm.]}\\
&= (a + (b + n(b))) + n(a) && \text{[by assoc.]}\\
&= (a+0) + n(a) && \text{[by inv. elem.]}\\
&= a + n(a) && \text{[by identity]}\\
&= 0 && \text{[by inv. elem.].}
\end{aligned}$$

Hence, $n(a+b) = n(a) + n(b)$.

□

(b) $q(q(a)) = a$

Proof. We need to show that $a \cdot q(a) = 1$. But that holds by definition of $q(a)$. Hence $q(q(a)) = a$.

□

9) Let F be an ordered field. Using only the order axioms show that if $0 < a < b$ and $0 < c < d$, then $ac < bd$.

Proof. Since $a > 0$, we have that $ac < ad$ [by multiplicativity]. Since $d > 0$, we have that $ad < db$. By transitivity, we have that $ac < db$.

□

10) Let $x \in \mathbb{R}$ such that $|x-1| < 2$. Show that $|x^2 - 2x + 2| < 5$.

Proof. We have:

$$\begin{aligned}
\left|x^2 - 2x + 2\right| &= \left|(x-1)^2 + 1\right| \\
&\leq \left|(x-1)^2\right| + |1| && \text{[by triang. ineq.]}\\
&= |x-1|^2 + 1 && \text{[as } |a|\,|b| = |ab|\text{]}\\
&< 2^2 + 1 = 5 && \text{[since } |x-1| < 2\text{].}
\end{aligned}$$

□