1) [20 points] Mark true or false. Justify your answers only for the ones which are false. [No need to justify if true.]

(a) $\mathcal{P} = \{\{1, 2, 3, 4, 5\}\}$ is a partition of $\{1, 2, 3, 4, 5\}$.

Answer: T.

(b) $\mathcal{P} = \{\{1\}, \{2, 5\}, \{3, 4\}\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$.

Answer: F, as 6 is not in any of the sets in the partition.

(c) Let X be the set of all people in the world, and define the relation $\mathcal{R}$ by $x\mathcal{R}y$ if x is married to y. Then, $\mathcal{R}$ is an equivalence relation.

Answer: F, as a person who is married is not married to herself/himself.

(d) Let X be the set of all people in the world, and define the relation $\mathcal{R}$ by $x\mathcal{R}y$ if x and y have the same mother. Then, $\mathcal{R}$ is an equivalence relation.

Answer: T.

(e) The converse of an "if-then" statement is logically equivalent to the original statement [i.e., either both are false or both are true].

Answer: F, for example "if $x = 1$, then $x > 0$" is true, but the converse "if $x > 0$, then $x = 1$" is false.

2) [20 points] Give the completely simplified negation of the following statement. [Your answers should have no "nots" in them.]

For all $\epsilon > 0$*, there is* $\delta > 0$ *such that either* $|x| \geq \delta$ *or* $|x^2| < \epsilon$*.*

[**Hint:** Negate one part at a time, as it makes it easier to get partial credit.]

Solution.

$$\text{NOT(For all } \epsilon > 0\text{, there is } \delta > 0 \text{ such that either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$$
$$\text{There exists } \epsilon > 0 \text{ such that NOT(there is } \delta > 0 \text{ such that either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$$
$$\text{There exists } \epsilon > 0 \text{ such that for all } \delta > 0\text{, NOT(either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$$
$$\text{There exists } \epsilon > 0 \text{ such that for all } \delta > 0\text{, } |x| < \delta \text{ and } |x^2| \geq \epsilon.$$

□

3) [20 points] Prove that if A and B are symmetric subsets of $\mathbb{R}$, then so is $A \cap B$.

Proof. As seen in class, a set X is symmetric if, and only if, for all $x \in X$, we have that $-x \in X$.
So, let $x \in A \cap B$. Then, by definition of intersection, $x \in A$ and $x \in B$. Since A and B are symmetric, we have that $-x \in A$ and $-x \in B$. Hence, by definition of intersection again, we have that $-x \in A \cap B$. Thus, $A \cap B$ is symmetric. □

4) [20 points] Let $\mathcal{R}$ be the relation on $\mathbb{R}^2$ given by $(x_1, y_1)\mathcal{R}(x_2, y_2)$ if there exists $r \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$.

[So, be careful here! You will use ordered pairs for elements! I.e., use "assume $(x_1, y_1)\mathcal{R}(x_2, y_2)$" instead of "assume $x\mathcal{R}y$" in your proofs.]

(a) Prove that $\mathcal{R}$ is an equivalence relation.

Proof. [Reflexive:] Given $(x, y) \in \mathbb{R}^2$, we have that $(x, y) = (x + 0, y + 0)$, and hence $(x, y)\mathcal{R}(x, y)$ [since $0 \in \mathbb{R}$].

[Symmetric:] Suppose that $(x_1, y_1)\mathcal{R}(x_2, y_2)$. Then, there is $r \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$. Thus, $(x_1, y_1) = (x_2 - r, y_2 - r)$. Since $-r \in \mathbb{R}$ [as $r, -1 \in \mathbb{R}$ and $\mathbb{R}$ is closed under multiplication], we have that $(x_2, y_2)\mathcal{R}(x_1, y_1)$.

[Transitive:] Suppose that $(x_1, y_1)\mathcal{R}(x_2, y_2)$ and $(x_2, y_2)\mathcal{R}(x_3, y_3)$. Then, by definition, there are $r, s \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$ and $(x_3, y_3) = (x_2 + s, y_2 + s)$. Hence, $(x_3, y_3) = (x_1 + (r + s), y_1 + (r + s))$, and since $r, s \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition, we have that $(x_1, y_1)\mathcal{R}(x_3, y_3)$.

□

(b) Give the equivalence class of $(0, 0)$.

Solution. We have:

$$\begin{aligned}\overline{(0,0)} &= \{(x, y) \in \mathbb{R}^2 \ : \ (0, 0)\mathcal{R}(x, y)\} \\ &= \{(x, y) \in \mathbb{R}^2 \ : \ \exists r \in \mathbb{R} \text{ such that } (x, y) = (0 + r, 0 + r)\} \\ &= \{(r, r) \ : \ r \in \mathbb{R}\}.\end{aligned}$$

Hence, $\overline{(0,0)}$ is the line $y = x$.

□

5) [20 points]

(a) Prove that $x \notin A \setminus C$ can be interpreted as saying that $x \notin A$ or $x \in C$. [You need to give a *formal* proof! Wordy arguments will receive partial credit at best. **Hint:** Remember how to negate "and/or" statements.]

Proof. Observe that to say that $x \notin A \setminus C$, is to negate $x \in A \setminus C$, i.e., to negate ($x \in A$ and $x \notin C$), which means that $x \notin A$ or $x \in C$ by De Morgan's Law.

[One can also use the contrapositive to prove this.]

□

(b) Let A, B and C be sets. Prove that $(B \cap C) \cup (B \setminus A) = B \setminus (A \setminus C)$. [**Hint:** Use item (a). Note that you can use it even if you did not do that part!]

Proof. [$\subseteq$:] Let $x \in (B \cap C) \cup (B \setminus A)$. Then, by definition of union, $x \in (B \cap C)$ or $x \in B \setminus A$. Then, by definition of intersection and complement, we have that ($x \in B$ and $x \in C$) or ($x \in B$ and $x \notin A$).
Thus, we have that $x \in B$ and (either $x \in C$ or $x \notin A$). Therefore, $x \in B$ and $x \notin A \setminus C$ [from part (a)]. By definition of complement, $x \in B \setminus (A \setminus C)$.

[$\supseteq$:] Let $x \in B \setminus (A \setminus C)$. Then, $x \in B$ and $x \notin A \setminus C$. This then means that $x \in B$ and (either $x \notin A$ or $x \in C$) by part (a). So, either ($x \in B$ and $x \notin A$) or ($x \in B$ and $x \in C$). Thus, $x \in B \setminus A$ or $x \in B \cap C$. Therefore, $x \in (B \cap C) \cup (B \setminus A)$.

□