1) [20 points] Mark true or false. Justify your answers only for the ones which are false. [No need to justify if true.]

(a) For every function $f : X \to Y$ and $A, B \subseteq X$, we have that $f(A \cap B) = f(A) \cap f(B)$.

Solution. False, for instance $f(x) = x^2$, $A = [-1, 0]$, $B = [0, 1]$, then $f(A) = f(B) = [0, 1]$, $f(A \setminus B) = f([-1, 0)) = (0, 1]$, while, $f(A) \setminus f(B) = \varnothing$. $\square$

(b) For every function $f : X \to Y$ and $A, B \subseteq Y$, we have that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution. T. $\square$

(c) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = 1/x^2$ is one-to-one.

Solution. False, as $f(-1) = f(1) = 1$. $\square$

(d) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = 1/x^2$ is onto.

Solution. False, as $f(x) > 0$ for all x, and hence there is no $x_0 \in \mathbb{R} \setminus \{0\}$ such that $f(a) = -1$. $\square$

(e) If $f : X \to Y$ is invertible and $A, B \subseteq X$, then both $f^{-1}(f(A)) = A$ and $f(A \setminus B) = f(A) \setminus f(B)$ are true.

Solution. T. $\square$

2) [20 points] Functions:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Give $f([-2, 3])$ and $f^{-1}((-1, 3))$.

Solution. Draw the picture! We have that $f([-2, 3]) = [0, 9]$, and $f^{-1}((-1, 3)) = (-\sqrt{3}, \sqrt{3})$. □

(b) Is $g : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $g(x) = 1/x$ invertible? [Don't forget to justify!]

Solution. Yes. Suffices to show that there is a function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ such that $g \circ h(x) = x$ and $h \circ g(x) = x$. But $g(x)$ is such function: $g \circ g(x) = g(g(x)) = 1/g(x) = 1/(1/x) = x$. □

3) [20 points] Prove *by induction* that for $n \in \mathbb{N}$:

$$\sum_{k=1}^{n}(2k+1) = n^2 + 2n.$$

Proof. We first prove for $n = 1$. Indeed, we have

$$\sum_{k=1}^{1}(2k+1) = 2 \cdot 1 + 1 = 3 = 1^2 + 2 \cdot 1.$$

Now suppose that

$$\sum_{k=1}^{m}(2k+1) = m^2 + 2m,$$

for some $m \geq 1$. [We need to prove that $\sum_{k=1}^{m+1}(2k+1) = (m+1)^2 + 2(m+1) = m^2+4m+3$.] We have:

$$\begin{aligned}\sum_{k=1}^{m+1}(2k+1) &= \left[\sum_{k=1}^{m}(2k+1)\right] + 2(m+1) + 1 \\ &= (m^2 + 2m) + 2m + 3 \\ &= m^2 + 4m + 3 \\ &= (m+1)^2 + 2(m+1).\end{aligned}$$

□

4) [20 points] Prove that $3^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Proof. We prove it by induction on n. For $n = 1$, we have that $3^2 - 1 = 8$ is divisible by 8. Now suppose that $3^{2m} - 1 = 8q$ for some $q \in \mathbb{Z}$ [i.e., $3^{2m} - 1$ is divisible by 8] for some $m \geq 1$. [We need to show that $3^{2(m+1)} - 1$ is divisible by 8.] Then,

$$\begin{aligned}
3^{2(m+1)} - 1 &= 3^{2m+2} - 1 \\
&= 9 \cdot 3^{2m} - 1 \\
&= (8+1)3^{2m} - 1 \\
&= 8 \cdot 3^{2m} + (3^{2m} - 1) \\
&= 8 \cdot 3^{2m} + 8q \\
&= 8 \cdot (3^{2m} + q).
\end{aligned}$$

Hence, since $q + 3^{2m} \in \mathbb{Z}$, 8 divides $3^{2(m+1)} - 1$.

□

5) [20 points] Prove that $n + 2 \leq 3^n$ for all integers $n \in \mathbb{N}$.

Proof. We prove it by induction on n again. For $n = 1$, we have $1 + 2 = 3 \leq 3 = 3^1$.
Now suppose that $m + 2 \leq 3^m$ for some $m \geq 1$. [We need to show that $(m + 1) + 2 \leq 3^{m+1}$.
We have

$$
\begin{aligned}
(m+1)+2 &= (m+2)+1 \\
&\leq 3^m + 1 && \text{[by the IH]} \\
&\leq 3^m + 3^m && \text{[as } 0 < m \text{ implies } 1 = 3^0 < 3^m] \\
&= 2 \cdot 3^m \\
&\leq 3 \cdot 3^m = 3^{m+1} && \text{[as } 2 < 3].
\end{aligned}
$$

Hence, $(m + 1) + 2 \leq 3^{m+1}$.

□