December 9th, 2009

Math 300

Luís Finotti
Spring 2009

Name:...

Student ID (last 6 digits): XXX- ....

Final

You have two hours to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 9 questions and 12 printed pages (including this one and two pages for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	10	
2	10	
3	10	
4	10	
5	12	
6	12	
7	12	
8	12	
9	12	
Total	100	

1) What's the coefficient of x^{20} in $(2+3x^4)^{100}$? [You do *not* need to evaluate powers and binomials.]

2) [Remember: if $a, b \in \mathbb{Z}$, then a divides b if there exists $q \in \mathbb{Z}$ such that $b = a \cdot q$.] Let $a, b, d \in \mathbb{Z}$. Prove that d divides a and b if, and only if, d divides a and $a + b$.

3) Prove or disprove: $A \setminus (B \cap C) = (A \setminus C) \cup (C \setminus B)$.

4) Let $\mathcal{R}$ be the relation on $\mathbb{R}$ given by $a\mathcal{R}b$ iff $a - b \in \mathbb{Z}$.

(a) Prove that $\mathcal{R}$ is an equivalence relation.

(b) Give three elements in the equivalence class $\overline{0.312}$, at least one of which is negative, and three elements *not* in $\overline{0.312}$, at least one of which is negative. [No need to justify this part.]

5) Find a closed formula for the recursion $a_0 = 0$, $a_n = 2 \cdot a_{n-1} - 3$ for $n \geq 1$. [You don't have to show me how you came up with the formula, but you have to prove that it is correct.]

6) Let $f : X \to Y$ and $A \subseteq Y$.

(a) Prove that if f is onto, then $f(f^{-1}(A)) = A$.

(b) Give an example of f and A such that $f(f^{-1}(A)) \neq A$.

7) Prove *by induction* that $\dfrac{n}{n+1} \geq \dfrac{1}{2}$ for all $n \in \mathbb{N}$. You can use any property of inequalities we've seen before, *as long as you state it clearly*!

[**Hint:** Prove first that $(n+1)^2 > n(n+2)$. [You do *not* need induction for that!] Then, note that $\dfrac{n+1}{n+2} = \dfrac{n}{n+1} \cdot \dfrac{(n+1)^2}{n(n+2)}$.]

8) Suppose that a and b are elements of an ordered field [you can think of $\mathbb{R}$ if you want] that have n-th roots, and $0 < a < b$. Prove that for all $n \in \mathbb{N}$ we have that $a^{1/n} < b^{1/n}$. [This is straight from your HW! You can use anything we've proved in class or HW about inequalities with *integer* exponents, *as long as you state it clearly*!]

9) Let F be a field. [Remember that if $a \in F$, then $n(a) = n(1) \cdot n(a)$, $n(n(a)) = a$, and if $a, b \in F \setminus \{0\}$, then $q(a \cdot b) = q(a) \cdot q(b)$. You can use those, without proving them, in both parts below.]

(a) Prove that $q(n(1)) = n(1)$. [**Hint:** Use that if $x \cdot a = 1$, then $x = q(a)$.]

(b) Prove that if $a \in F \setminus \{0\}$, then $q(n(a)) = n(q(a))$. [**Hint:** It might help to use (a).]

Scratch:

Scratch: