1) What's the coefficient of x^{20} in $(2+3x^4)^{100}$? [You do *not* need to evaluate powers and binomials.]

Solution. We have

$$(2+3x^4)^{100} = \sum_{i=0}^{100} \binom{100}{i} (3x^4)^i 2^{100-i}.$$

Hence, the coefficient is $\binom{100}{5} 3^5 2^{95}$ [i.e., we take $i=5$]. □

2) [Remember: if $a, b \in \mathbb{Z}$, then a divides b if there exists $q \in \mathbb{Z}$ such that $b = a \cdot q$.] Let $a, b, d \in \mathbb{Z}$. Prove that d divides a and b if, and only if, d divides a and $a + b$.

Proof. [$\Rightarrow$] Suppose that d divides a and b. [We need to show that d divides a and $a + b$.] Then, by definition there $q_1, q_2 \in \mathbb{Z}$ such that $a = q_1 \cdot d$ and $b = q_2 \cdot d$. Then, $a + b = q_1 \cdot d + q_2 \cdot d = (q_1 + q_2) \cdot d$, and hence [since $\mathbb{Z}$ is closed under addition] d divides $a + b$ by definition [of division]. Since d also divides a [by assumption], we have that d divides a and $a + b$.

[$\Leftarrow$] Suppose now that d divides a and $a + b$. [We need to show that d divides a and b.] Then, by definition, there are $q_1, q_3 \in \mathbb{Z}$ such that $a = q_1 \cdot d$ and $a + b = q_3 \cdot d$. Hence, $b = (a + b) - b = q_3 \cdot d - q_1 \cdot d = (q_3 - q_1) \cdot d$, and thus [since $\mathbb{Z}$ is closed under subtraction] we have that d divides b by definition [of division]. Since d also divides a [by assumption], we have that d divides a and b.

$\square$

3) Prove or disprove: $A \setminus (B \cap C) = (A \setminus C) \cup (C \setminus B)$.

Solution. The statement is *false*! [Again, it suffices to give a counterexample.] Let $A = B = \varnothing$ and $C = \{1\}$. Then, $A \setminus (B \cap C) = \varnothing$. Also, $A \setminus C = \varnothing$ and $C \setminus B = \{1\}$. Hence $(A \setminus C) \cup (C \setminus B) = \{1\} \neq \varnothing = A \setminus (B \cap C)$. □

4) Let $\mathcal{R}$ be the relation on $\mathbb{R}$ given by $a\mathcal{R}b$ iff $a - b \in \mathbb{Z}$.

(a) Prove that $\mathcal{R}$ is an equivalence relation.

Proof. [Reflexive:] [We need to prove that $x\mathcal{R}x$ for all $x \in \mathbb{R}$.] Given $x \in \mathbb{R}$, we have that $x - x = 0 \in \mathbb{Z}$. Thus, $x\mathcal{R}x$ [by definition].

[Symmetric:] Suppose that $x\mathcal{R}y$. [We need to prove that $y\mathcal{R}x$.] Then, [by definition] we have that $x - y \in \mathbb{Z}$. Thus, $-(x - y) = y - x \in \mathbb{Z}$ [as $0 \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under subtraction]. Hence, $y\mathcal{R}x$ [by definition].

[Transitive:] Suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$. [We need to prove that $x\mathcal{R}z$.] By definition, we have that $x - y, y - z \in \mathbb{Z}$. Hence, [since $\mathbb{Z}$ is closed under addition] we have that $(x - y) + (y - z) = x - z \in \mathbb{Z}$, and thus $x\mathcal{R}z$ [by definition].

□

(b) Give three elements in the equivalence class $\overline{0.312}$, at least one of which is negative, and three elements *not* in $\overline{0.312}$, at least one of which is negative. [No need to justify this part.]

Solution. We have that $0.312, 1.312, \underbrace{0.312 - 1}_{=-0.688} \in \overline{0.312}$, and $-1, 0, 1 \notin \overline{0.312}$. □

5) Find a closed formula for the recursion $a_0 = 0$, $a_n = 2 \cdot a_{n-1} - 3$ for $n \geq 1$. [You don't have to show me how you came up with the formula, but you have to prove that it is correct.]

Solution. We have

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= -3 \\
a_2 &= 2 \cdot (-3) + (-3) \\
a_3 &= 4 \cdot (-3) + 2 \cdot (-3) + (-3) \\
a_4 &= 8 \cdot (-3) + 4 \cdot (-3) + 2 \cdot (-3) + (-3) \\
&\vdots \\
a_n &= 2^{n-1} \cdot (-3) + 2^{n-2} \cdot (-3) + \cdots + 2^1 \cdot (-3) + 2^0 \cdot (-3) \\
&= (-3) \cdot (2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0) \\
&= -3 \cdot \frac{2^n - 1}{2 - 1} = -3 \cdot (2^n - 1).
\end{aligned}$$

So, we claim that $a_n = -3 \cdot (2^n - 1)$, and prove it by induction.

For $n = 0$, we have that $a_0 = 0 = -3 \cdot (2^0 - 1)$.

Now, suppose that $a_n = -3 \cdot (2^n - 1)$. [We need to prove that $a_{n+1} = -3 \cdot (2^{n+1} - 1)$.] We then have:

$$\begin{aligned}
a_{n+1} &= 2 \cdot a_n - 3 && \text{[recurrence]} \\
&= 2 \cdot (-3 \cdot (2^n - 1)) - 3 && \text{[ind. hyp.]} \\
&= -3 \cdot (2 \cdot (2^n - 1) + 1) && \text{[factor } -3\text{]} \\
&= -3 \cdot (2^{n+1} - 2 + 1) = -3 \cdot (2^{n+1} - 1).
\end{aligned}$$

□

6) Let $f : X \to Y$ and $A \subseteq Y$.

(a) Prove that if f is onto, then $f(f^{-1}(A)) = A$.

Proof. [$\subseteq$] Let $y \in f(f^{-1}(A))$. [We need to show that $y \in A$.] Then, by definition of direct image, there exists $x \in f^{-1}(A)$ such that $y = f(x)$. But, by definition of preimage, we have that $x \in f^{-1}(A)$ means that $f(x) \in A$. Since $y = f(x)$, we have that $y \in A$. [Note that we did not use the fact that f is onto here.]

[$\supseteq$] Let $y \in A$. [We need to show that $y \in f(f^{-1}(A))$.] Since f is onto, there exists $x \in X$ such that $y = f(x)$. Since $y \in A$, by definition of preimage, we have that $x \in f^{-1}(A)$. Since $y = f(x)$ and $x \in f^{-1}(A)$, by definition of direct image, $y \in f(f^{-1}(A))$. [Note that the fact f is onto is used in this part.]

□

(b) Give an example of f and A such that $f(f^{-1}(A)) \neq A$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, and take $A = \{-1, 1\}$. Then, $f^{-1}(A) = \{-1, 1\}$, and thus $f(f^{-1}(A)) = \{1\} \neq \{-1, 1\}$.

□

7) Prove *by induction* that $\dfrac{n}{n+1} \geq \dfrac{1}{2}$ for all $n \in \mathbb{N}$. You can use any property of inequalities we've seen before, *as long as you state it clearly*!

[**Hint:** Prove first that $(n+1)^2 > n(n+2)$. [You do *not* need induction for that!] Then, note that $\dfrac{n+1}{n+2} = \dfrac{n}{n+1} \cdot \dfrac{(n+1)^2}{n(n+2)}$.]

Proof. First, observe that $(n+1)^2 = n^2+2n+1 > n^2+2n = n(n+2)$. Then, for $n \neq -1, -2$, we have that $\dfrac{(n+1)^2}{n(n+2)} > 1$.

Now, we prove the statement by induction. For $n = 1$, we have $1/(1+1) \geq 1/2$.

Suppose then that $\dfrac{n}{n+1} \geq \dfrac{1}{2}$ for some $n \geq 1$. [We need to prove that $\dfrac{n+1}{n+2} \geq \dfrac{1}{2}$.] As observed above, since $n \neq -1, -2$, we have that $\dfrac{(n+1)^2}{n(n+2)} > 1$, and then

$$\frac{n+1}{n+2} = \frac{n}{n+1} \cdot \frac{(n+1)^2}{n(n+2)} \geq \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

[Here, we've used the fact that if $0 < a \leq b$ and $0 < c \leq d$, then $ac \leq bd$.]

□

8) Suppose that a and b are elements of an ordered field [you can think of $\mathbb{R}$ if you want] that have n-th roots, and $0 < a < b$. Prove that for all $n \in \mathbb{N}$ we have that $a^{1/n} < b^{1/n}$. [This is straight from your HW! You can use anything we've proved in class or HW about inequalities with *integer* exponents, *as long as you state it clearly*!]

Proof. We prove the result by contradiction. Suppose that $a^{1/n} \geq b^{1/n}$. [We must derive a contradiction.]

If $a^{1/n} = b^{1/n}$, then $a = (a^{1/n})^n = (b^{1/n})^n = b$, which is a contradiction [as $a < b$].

If $a^{1/n} > b^{1/n}$, since we know $b^{1/n} > 0$ [by definition of n-th root], we have that $a = (a^{1/n})^n > (b^{1/n})^n = b$ [as if $0 < x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$], which again contradicts $a < b$.

□

9) Let F be a field. [Remember that if $a \in F$, then $n(a) = n(1) \cdot n(a)$, $n(n(a)) = a$, and if $a, b \in F \setminus \{0\}$, then $q(a \cdot b) = q(a) \cdot q(b)$. You can use those, without proving them, in both parts below.]

(a) Prove that $q(n(1)) = n(1)$. [**Hint:** Use that if $x \cdot a = 1$, then $x = q(a)$.]

Proof. We have that $n(1) \cdot n(1) = n(n(1)) = 1$. As stated in the hint, this means that $n(1) = q(n(1))$. □

(b) Prove that if $a \in F \setminus \{0\}$, then $q(n(a)) = n(q(a))$. [**Hint:** It might help to use (a).]

Proof. We have

$$q(n(a)) = q(n(1) \cdot a) = q(n(1)) \cdot q(a) = n(1) \cdot q(a) = n(q(a)).$$

□