December 9th, 2009

Math 351

Luís Finotti
Spring 2009

Name:. . . . .

Student ID (last 6 digits): XXX- . . . .

Final

You have two hours to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 8 questions and 11 printed pages (including this one and two pages for scratch work in the end).

No books or notes are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	20	
Total	100	

1) Let $\sigma, \tau \in S_8$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 2 & 8 & 6 & 7 \end{pmatrix} \quad \text{and} \quad \tau = (1\;4\;2\;5)(3\;6\;7).$$

(a) Write the complete factorization of σ into disjoint cycles.

(b) Compute σ^{-1}, and τ^{-1}. [Your answer can be in any form.]

(c) Compute $\tau\sigma$. [Your answer can be in any form.]

(d) Compute $\tau^{-1}\sigma\tau$. [Your answer can be in any form.]

(e) Write τ as a product of transpositions.

2) Give all *possible* rational roots of

$$f(x) = x^5 + \frac{2}{3}x^4 - 2x^3 + 7x^2 - x + 1.$$

You do not have to find which ones give roots, just list the possible rational roots! [**Hint:** Be careful! Don't be tricked!]

3) Let $f(x) = x^5 + 1$ and $g(x) = x^3 + 1$ in $\mathbb{F}_2[x]$. Write the GCD of f and g as a linear combination of them.

4) Determine which of the following polynomials are irreducible in $\mathbb{Q}[x]$. [Justify!]

(a) $f(x) = x^3 - 3x^2 + 2x - 7$.

(b) $f(x) = x^4 + 1$. [**Hint:** What happens with $f(x+1)$?]

(c) $f(x) = 3x^7 + 6x^4 + 81x^3 - 9x + 1$ [**Hint:** Using a [tricky] HW problem makes this much easier!]

5) Let F be a field and $f, g \in F[x]$. Let also

$$I = \{f \cdot r + g \cdot s \ : \ r, s \in F[x]\}.$$

[Hence, I is a the set of all linear combinations of f and g.] Prove that there exist $d \in F[x]$ such that

$$I = \{d \cdot t \ : \ t \in F[x]\}.$$

[**Hint:** d is the GCD of f and g. Also, we've done the analogue of this for integers in class! The proof is the same.]

6) Give example polynomials $f, g \in R[x]$, for some suitable ring R, such that f has more [distinct] roots in R than its degree, and g has degree greater than zero and yet is a unit. [**Hint:** Take $R = \mathbb{Z}/n\mathbb{Z}$ for the smallest $n > 1$ for which R is not a domain. The degrees of f and g can be low. Note that I showed you these examples in class!]

7) Prove that there is no integer n whose square n^2 has its last two digits as 35. [**Hint:** If the last digit of n^2 is 5, what can we say about the last digit of n, i.e., what is the remainder of n when divided by 10? Then, what happens with n^2 modulo 100?]

8) Let F be a field with exactly 4 elements, say $F = \{0, 1, a, b\}$. [Hence, we are assuming that all these elements are distinct, e.g., $a \neq 1$, $b \neq 0$, etc.]

(a) Prove that $1 = -1$ in F. [**Hint:** Suppose not. Then, $-1 \neq 1$. Then, as $-1 \neq 0$, we can assume without loss of generality, that $-1 = a$. Show then that $b = -b$ by checking that no other element can be $-b$. This would mean that $b + b = b(1 + 1) = 0$. Since $b \neq 0$ and we are in a field, we would have that $1 + 1 = 0$, contradicting the assumption that $1 \neq -1$.]

(b) Prove that $b = a + 1$. [**Hint:** Can $a + 1$ be any other element? You need to use (a)!]

(c) Prove that if $b = a^2$. [**Hint:** Can a^2 be any other element? You need to use (a) and the fact that $xy = 0$ implies that either $x = 0$ or $y = 0$.]

(d) Prove that a is a root of $x^2 + x + 1 \in F[x]$. [Use the previous items.]

Scratch:

Scratch: