1) *Quickies!* You don't need to justify your answers.

(a) If the reduced row echelon form of a square matrix A is *not* the identity matrix, what can you say about the number of solutions of $A\mathbf{x} = \mathbf{0}$?

Solution. It has infinitely many solutions. $\square$

(b) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Compute $A \cdot E$.

Solution. Since E is obtained from I_3 by adding -2 times the first column to the second, we have that $A \cdot E$ is obtained by doing the same column operation to A:

$$A \cdot E = \begin{bmatrix} 2 & -4 & 1 \\ -1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

$\square$

(c) Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, compute $\begin{vmatrix} 2d & 2e & 2f \\ a & b & c \\ g-3a & h-3b & i-3c \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 2d & 2e & 2f \\ a & b & c \\ g-3a & h-3b & i-3c \end{vmatrix} = \begin{vmatrix} 2d & 2e & 2f \\ a & b & c \\ g & h & i \end{vmatrix} =$$

$$2 \cdot \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = 2 \cdot (-1) \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2 \cdot (-1) \cdot 3 = -6.$$

□

(d) Give the equation of a plane perpendicular to $x + y - 2z = 3$.

Solution. The vector $\mathbf{v} = (1, 1, -2)$ is perpendicular to the plane. There are many choices of vectors perpendicular to $\mathbf{v}$, for instance $(1, -1, 0)$, or $(2, 0, 1)$, or $(1, 1, 1)$, etc. So, for example, $x - y = 0$ [or $2x + z = 3$, or $x + y + z = 1$] is perpendicular to the given plane. □

(e) If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, find A such that $D \cdot A = B$.

Solution. We have

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix},$$

and so,

$$A = D^{-1}B = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -2 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

□

(f) Let $\mathbf{v} = (1, 2)$ and $\mathbf{a} = (1, 1)$. Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, with $\mathbf{v}_1$ having the same direction as $\mathbf{a}$ and $\mathbf{v}_2$ being perpendicular to $\mathbf{a}$.

Solution. We have:

$$\mathbf{v}_1 = \operatorname{proj}_{\mathbf{a}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|^2} \mathbf{a} = \frac{3}{2}(1, 1) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Then,

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (1, 2) - \left(\frac{3}{2}, \frac{3}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

□

(g) Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$. Compute $(A \cdot B)^{\mathrm{T}}$.

Solution. We have

$$A \cdot B = \begin{bmatrix} 9 & -2 & 1 \\ 7 & -1 & 1 \end{bmatrix},$$

so,

$$(A \cdot B)^{\mathrm{T}} = \begin{bmatrix} 9 & 7 \\ -2 & -1 \\ 1 & 1 \end{bmatrix}.$$

□

(h) For what values of k is $A = \begin{bmatrix} 1 & -2 & 3 & 0 & 1 \\ 0 & k & 2 & -k & 3 \\ 0 & 0 & (k-1) & k^2 & 0 \\ 0 & 0 & 0 & (k+1)^2 & k \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ invertible?

Solution. The elements of the main diagonal cannot be zero, so it is invertible for all values of k except $k = 0, 1, -1$. □

2) Let

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ -2 & 1 & 5 & 1 \\ -3 & 2 & 2 & -1 \\ 4 & -3 & 1 & 4 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find all solutions [if any] of the systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$.

Solution. We solve the systems simultaneously:

$$\left[\begin{array}{rrrr|r|r} 1 & -1 & 3 & 2 & 1 & 2 \\ -2 & 1 & 5 & 1 & 1 & 0 \\ -3 & 2 & 2 & -1 & 0 & 1 \\ 4 & -3 & 1 & 4 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrrr|r|r} 1 & -1 & 3 & 2 & 1 & 2 \\ 0 & -1 & 11 & 5 & 3 & 4 \\ 0 & -1 & 11 & 5 & 3 & 7 \\ 0 & 1 & -11 & -4 & -5 & -7 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrr|r|r} 1 & -1 & 3 & 2 & 1 & 2 \\ 0 & 1 & -11 & -5 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{array}\right] \sim \left[\begin{array}{rrrr|r|r} 1 & 0 & -8 & -3 & -2 & -2 \\ 0 & 1 & -11 & -5 & -3 & -4 \\ 0 & 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r|r} 1 & 0 & -8 & 0 & -8 & -11 \\ 0 & 1 & -11 & 0 & -13 & -19 \\ 0 & 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

Hence, the first system has solution $x_1 = -8 + 8t$, $x_2 = -13 + 11t$, $x_3 = t$, and $x_4 = -2$, for $t \in \mathbb{R}$, while the second system has no solution.

□

3) Let $A = \begin{bmatrix} 1 & 3 & 0 & 5 & 1 \\ 1 & 2 & 2 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 1 & 1 \\ 2 & 6 & 1 & 1 & 2 \end{bmatrix}$. Compute $\det(-A^3)$.

Solution. We have:

$$\begin{aligned}
\det(A) &= (-1)^{3+4} \begin{vmatrix} 1 & 3 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 3 & -1 & 0 & 1 \\ 2 & 6 & 1 & 2 \end{vmatrix} && \text{[use 3rd row]} \\
&= -1 \cdot \left((-1)^{2+3} \cdot 2 \cdot \begin{vmatrix} 1 & 3 & 1 \\ 3 & -1 & 1 \\ 2 & 6 & 2 \end{vmatrix} + (-1)^{4+3} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & -1 & 1 \end{vmatrix} \right) && \text{[use 3rd col.]} \\
&= -1 \cdot (-1 \cdot 2 \cdot 0 + -1 \cdot (2 + 18 - 1 - (6 - 2 + 3))) && \text{[row mult. of another and Sarrus]} \\
&= 12.
\end{aligned}$$

So, $\det(-A^3) = (-1)^5 \det(A^3) = -(\det(A))^3 = -12^3 = -1728$.

□

4) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Given that A is invertible with $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$, find B^{-1} where $B = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

[**Note:** Observe that B is obtained from A by switching the first and second rows. You can use this and A^{-1} to compute B^{-1} in one second, *but you need to justify*! If you don't see it, or cannot justify, just compute B^{-1} directly.]

Solution. Let $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then, $E \cdot A = B$, as E is the elementary row obtained by switching the first and second rows of the identity.

Then, $B^{-1} = (E \cdot A)^{-1} = A^{-1} \cdot E^{-1}$. Now, $E^{-1} = E$, as $E \cdot E = I_3$, since it switches the first and second rows of E back to the identity.

So, $B^{-1} = A^{-1} \cdot E$. Since E is on the left, we have to see which *column* operations performed to I_3 gives us E. In this case it is switching the first and second *columns*.

Thus, $B^{-1} = A^{-1} \cdot E$ is obtained by switching the first and second *columns* of A^{-1}, in other words,

$$B^{-1} = \begin{bmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{bmatrix}.$$

□