Midterm Solution

M551 – Abstract Algebra

October 14th, 2011

1. Let G be a finite *simple* group with $|G| = p^\alpha n$, where $n > 1$, p is prime, $\alpha \in \mathbb{Z}_{>0}$ and $p \nmid n$. Show that $|G| \mid n_p!$,where n_p is the number of Sylow p-subgroups of G.

Proof. By Sylow's Theorem, we have that G acts on $\mathrm{Syl}_p(G)$ by conjugation. Since G is simple and $n > 1$, we have that $n_p > 1$. Also, the kernel of the representation of this action is either $\{1\}$ or G. Since the action is non-trivial, as it is transitive [by Sylow's Theorem] and $n_p > 1$, we must have that the kernel is $\{1\}$, and thus, by the first isomorphism theorem, G is isomorphic to a subgroup of S_{n_p}. Therefore, by Lagrange's Theorem, we have that $|G| \mid n_p!$. □

2. (a) Let G act on a finite set S and assume that there exists an element in G which induces an *odd* permutation of S. Show that there exists $H \leq G$ such that $|G : H| = 2$.

Proof. Let $|S| = n$. Then, the action gives a representation $\phi : G \to S_n$. Let $\epsilon : S_n \to \{\pm 1\}$ be the sign (or parity) homomorphism.

Since G has an element that induces an odd permutation [and $\epsilon(1) = 1$], we have that the homomorphism $\epsilon \circ \phi : G \to \{\pm 1\}$ is onto. Let H be its kernel. Then, by the First Isomorphism Theorem, we have that $|G : H| = |G| / |H| = |\{\pm 1\}| = 2$. □

(b) Let G be a finite group of order $2n$, where n is odd. Show that G has a subgroup of index 2. [**Hint:** Let G act on itself by left multiplication.]

Proof. We use part (a) with G acting on itself by left multiplication. We just need an element which induces an odd permutation. Let g be an element of order 2 [by Cauchy].

Note that the kernel of this action is trivial, as the only element of $x \in G$ such that $xy = y$ for all $y \in G$ is $x = 1$. So, G is isomorphic to a subgroup of S_{2n}. Also note that if $x \neq 1$, then $xy \neq y$ for all $y \in G$ [i.e., x fixes no element of G].

Since g has order 2, we have that if $g = \sigma_1 \cdots \sigma_t$, where the σ_i's are non-trivial *disjoint* cycles of S_{2n}, then the lcm of the length of these cycles is 2, i.e., all σ_i's are transpositions. Since g fixes no element, we must have that the σ_i's involve all $2n$ elements, i.e., $t = n$. Since n is odd, g induces an odd permutation. □