1) [15 points] Draw the graph of a function $y = f(x)$ such that:

(i) $\lim_{x \to -2} f(x) = 1$

(ii) $\lim_{x \to 0} f(x) = -\infty$

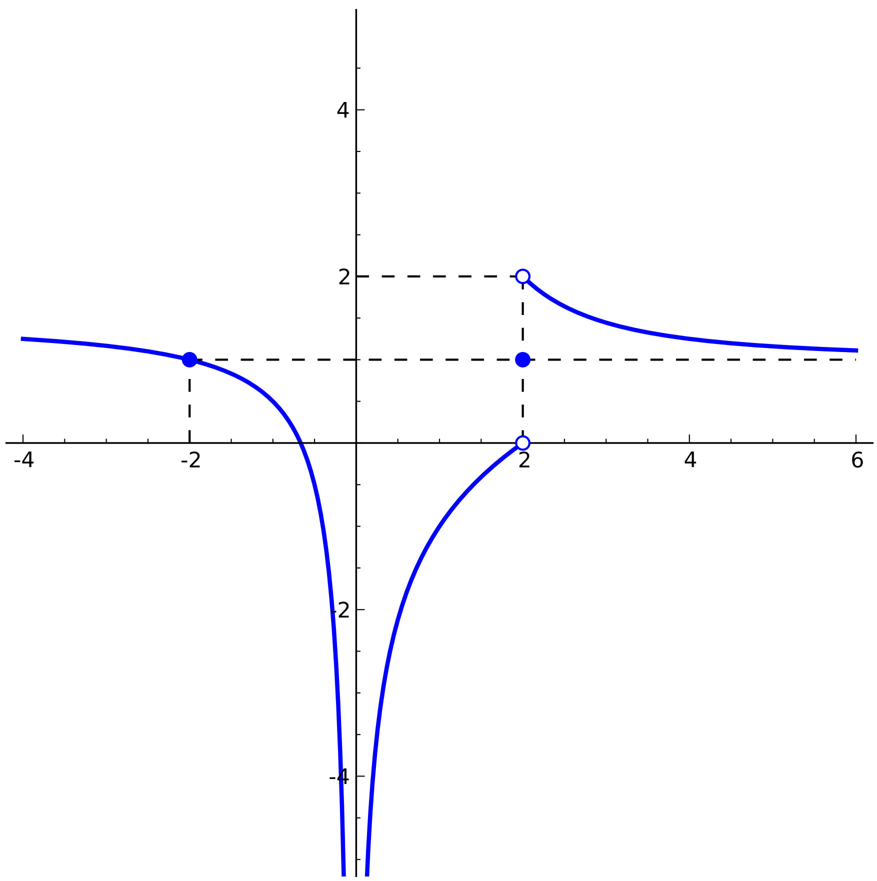
(iii) $\lim_{x \to 2^-} f(x) = 0$

(iv) $\lim_{x \to 2^+} f(x) = 2$

(v) $f(2) = 1$ [this is not a limit]

(vi) $\lim_{x \to \infty} f(x) = 1$

Solution. There are many ways to draw this. Here is one possibility:



$\square$

2) Compute the following limits.

(a) [6 points] $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4}$

Solution.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} &= \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{(x-2)}{(x+2)} \\
&= \frac{0}{4} = 0.
\end{aligned}$$

□

(b) [7 points] $\lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2}$

Solution.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} &= \lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} \cdot \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} \\
&= \lim_{x\to 2} \frac{(x-1)-1}{(x-2)(\sqrt{x-1}+1)} \\
&= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} = \frac{1}{2}.
\end{aligned}$$

□

(c) [6 points] $\lim_{x\to\infty} \frac{\sqrt{9x^4+x^2+1}}{x-2x^2}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{9x^4+x^2+1}}{x-2x^2} &= \lim_{x\to\infty} \frac{x^2\sqrt{9+x^{-2}+x^{-4}}}{x^2(x^{-1}-2)} \\
&= \lim_{x\to\infty} \frac{\sqrt{9+x^{-2}+x^{-4}}}{(x^{-1}-2)} \\
&= \frac{\sqrt{9}}{-2} = -\frac{3}{2}.
\end{aligned}$$

□

(d) [7 points] $\lim_{x\to 0} \frac{\sin(3x)\cos(4x)}{2x}$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x)\cos(4x)}{2x} &= \left(\lim_{x\to 0} \cos(4x)\right) \cdot \left(\lim_{x\to 0} \frac{\sin(3x)}{2x}\right) \\
&= 1 \cdot \frac{1}{2} \left(\lim_{x\to 0} \frac{\sin(3x)}{x}\right) \\
&= \frac{1}{2} \left(\lim_{u\to 0} \frac{\sin(u)}{u/3}\right) && \text{[subs. } u = 3x\text{]} \\
&= \frac{3}{2} \left(\lim_{u\to 0} \frac{\sin(u)}{u}\right) = \frac{3}{2}
\end{aligned}$$

□

3) [15 points] Compute the derivative of $f(x) = x + \dfrac{1}{x}$ *using limits*. [You *cannot* use formulas. You can check your result with the formulas, though.]

Solution. We have:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\left(x + h + \dfrac{1}{x+h}\right) - \left(x + \dfrac{1}{x}\right)}{h} \\
&= \lim_{h \to 0} \frac{h + \dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{h + \dfrac{x - (x+h)}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{h + \dfrac{-h}{x(x+h)}}{h} \\
&= \lim_{h \to 0} 1 + \frac{-1}{x(x+h)} \\
&= 1 - \frac{1}{x^2}.
\end{aligned}
$$

So, $f'(x) = 1 - \dfrac{1}{x^2}$. □

4) [14 points] Give the equation of the line tangent to the graph of $f(x) = 2\sqrt{x} - \dfrac{3}{x}$ at $x = 1$.

Solution. We have [using formulas] $f'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{3}{x^2}$. Then, $f'(1) = 1 + 3 = 4$. Since $f(1) = -1$, we have that the equation of the tangent line is:

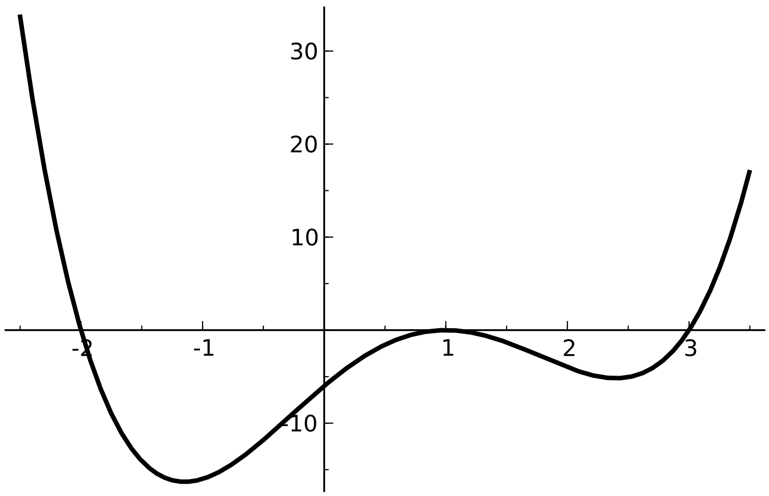
$$y + 1 = 4(x - 1).$$

$\square$

5) [15 points] Give a [finite] closed interval in which we have a solution to $2^x + 3^x = 4^x$. [Justify!]

Solution. Let $f(x) = 2^x + 3^x - 4^x$. Then, $f(1) = 1$ and $f(2) = -3$. So, by the *Intermediate Value Theorem*, there is a zero of $f(x)$, and therefore a solution of the equation, in the closed interval $[1, 2]$. $\square$

6) [15 points] The graph of $f(x)$ is given below.



Answer the following questions about the values of the *derivative* of $f(x)$. **No need to justify these.**

(a) Fill the table below with $+$, $-$, or 0 if the corresponding value is positive, negative, or zero respectively.

value:	$f'(-2)$	$f'(0)$	$f'(1)$	$f'(2)$	$f'(3)$
sign:	$-$	$+$	0	$-$	$+$

(b) Which is larger, $f'(-2)$ or $f'(2)$? [If they are equal, just say so.]

Solution. We have that $f'(2) > f'(-2)$, as the slope of the tangent line at $x = -2$ is "more negative" then the slope of the tangent line at $x = 2$. $\square$