1) Compute the derivatives of the following functions. [**No need to simplify your answers!**]

(a) [6 points] $f(x) = x \cdot \mathrm{e}^x \cdot \cos(x)$

Solution.

$$\begin{aligned} f'(x) &= 1 \cdot \mathrm{e}^x \cdot \cos(x) + x \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x \cdot \cos(x)) \\ &= \mathrm{e}^x \cdot \cos(x) + x(\mathrm{e}^x \cdot \cos(x) + \mathrm{e}^x \cdot (-\sin(x))) \\ &= \mathrm{e}^x(\cos(x) + x(\cos(x) - \sin(x))). \end{aligned}$$

□

(b) [7 points] $f(x) = \dfrac{\sin(\mathrm{e}^x + 1)}{2x^2}$

Solution.

$$\begin{aligned} f'(x) &= \frac{\cos(\mathrm{e}^x + 1) \cdot \mathrm{e}^x \cdot 2x^2 - \sin(\mathrm{e}^x + 1) \cdot 4x}{(2x^2)^2} \\ &= \frac{\cos(\mathrm{e}^x + 1) \cdot \mathrm{e}^x \cdot 2x^2 - \sin(\mathrm{e}^x + 1) \cdot 4x}{4x^4} \\ &= \frac{\cos(\mathrm{e}^x + 1) \cdot \mathrm{e}^x \cdot x - \sin(\mathrm{e}^x + 1) \cdot 2}{2x^3} \end{aligned}$$

□

(c) [7 points] $f(x) = \arctan(x)^x$. [**Note:** $\arctan(x)$ is the same as $\tan^{-1}(x)$.]

Solution. Using logarithmic derivative:

$$\begin{aligned} f'(x) &= f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\ln(f(x))) \\ &= \arctan(x)^x \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x \cdot \ln(\arctan(x))) \\ &= \arctan(x)^x \cdot \left(\ln(\arctan(x)) + x \frac{1}{\arctan(x)} \cdot \frac{1}{1 + x^2} \right) \end{aligned}$$

□

2) [15 points] Find the equation of the line tangent to the curve given by

$$x^2y = x - y^3 + 1$$

at the point $(0, 1)$.

Solution. Taking derivatives of both sides with respect to x, we get:

$$2xy + x^2y' = 1 - 3y^2y'.$$

Solving for y', we get:

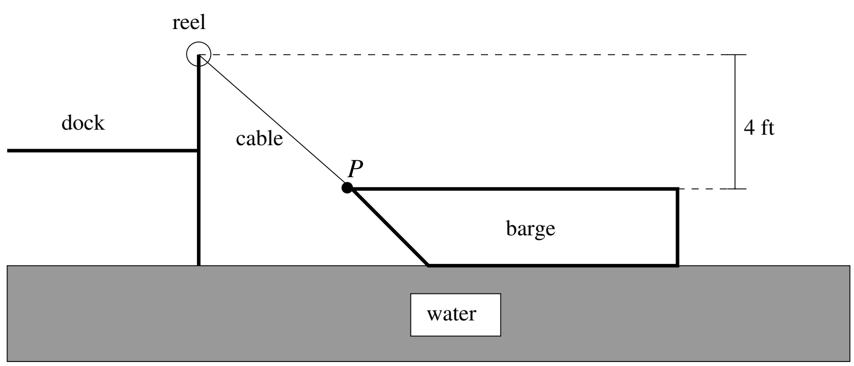
$$y' = \frac{1 - 2xy}{x^2 + 3y^2}.$$

So, the slope at the point $(0, 1)$ is $1/3$ and the equation of the tangent line is

$$y - 1 = \frac{1}{3}(x - 0).$$

□

3) [15 points] A machine on a dock reels a cable, attached to a barge [see picture below], with a speed of 1 foot per second. The point at which the cable is attached, labeled P in the picture, is 4 feet lower than the reel and is located at the very front of the barge. How fast is the barge moving when the front of it [point P] is 3 feet away [horizontally!] from the dock?



Solution. Let s be the horizontal distance between the barge and the dock and l be the length of the rope. Then, by Pythagoras, we have that

$$l^2 = s^2 + 4^2.$$

Taking derivatives with respect to time we get

$$2 \cdot l \cdot l' = 2 \cdot s \cdot s', \qquad \text{and so} \qquad s' = \frac{l \cdot l'}{s}.$$

We have that $l' = 1$ [from the statement] and when $s = 3$ we have that, by Pythagoras, $l = 5$ and so $s' = 5/3$, which is the speed at which the barge approaches the dock [in feet per second]. □

4) [15 points] Use the tangent line approximation to estimate $\sqrt{10}$.

Solution. Let $f(x) = \sqrt{x}$. Then, close to $x = 9$, the tangent line approximation gives us that:

$$f(x) = \sqrt{x} \approx f(9) + f'(9)(x-9) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) = 3 + \frac{1}{6}(x-9).$$

So,

$$\sqrt{10} \approx 3 + \frac{1}{6} = \frac{19}{6} = 3.3333\ldots.$$

□

5) [15 points] Find [absolute] maximum and minimum [*both* x-coordinate and corresponding value of the function] of $f(x) = 2x^3 - 3x^2 + 2$ in the interval $[-1, 2]$.

Solution. We have

$$f'(x) = 6x^2 - 6x = 6x(x-1)$$

So, the derivative is defined everywhere and it is zero at $x = 0$ and $x = 1$.

We have

$$\begin{aligned} f(-1) &= -3 \\ f(0) &= 2 \\ f(1) &= 1 \\ f(2) &= 6. \end{aligned}$$

So, the absolute maximum is 6 at $x = 2$ and the absolute minimum is -3 at $x = -1$. □

6) [20 points] Let $f(x) = x^{7/3} - 7x^{1/3} + 1$. Its first and second derivatives are

$$f'(x) = \frac{7}{3}(x^{4/3} - x^{-2/3}) \qquad \text{and} \qquad f''(x) = \frac{14}{9}(2x^{1/3} + x^{-5/3})$$

respectively. [No need to check! Just use it!]
[**Note:** In all items below you can use "DNE" for "does not exist".]

(a) Give the intervals in which $f(x)$ is increasing and the intervals in which it is decreasing.

Solution. It is increasing where $f'(x)$ is positive and decreasing where negative.

Solving where is zero:

$$\frac{7}{3}(x^{4/3} - x^{-2/3}) = \frac{7}{3}x^{-2/3}(x^2 - 1) = \frac{7}{3}x^{-2/3}(x-1)(x+1) = 0$$

So, it is zero at ± 1 and not defined at $x = 0$. Analyzing the signs, we get: [positive, so] increasing on $(-\infty, -1) \cup (1, \infty)$ and [negative, so] decreasing on $(-1, 0) \cup (0, 1)$. [Decreasing on $(-1, 1)$ is also OK.] □

(b) Give all critical points [x-coordinate only] and classify them as local maximum, local minimum or neither.

Solution. The x-coordinates of the critical points are -1 (local maximum), 0 (neither) and 1 (local minimum). □

(c) Give all intervals in which the graph of the function $f(x)$ is concave up and all intervals in which it is concave down.

Solution. It is concave up where $f''(x)$ is positive and down where negative.

Solving where is zero:

$$\frac{14}{9}(2x^{1/3} + x^{-5/3}) = \frac{14}{9}x^{-5/3}(2x^2 + 1) = 0$$

But this function is then never 0 and it is not defined at $x = 0$. Analyzing the signs: [negative, so] concave down on $(-\infty, 0)$ and [positive, so] concave up on $(0, \infty)$. □

(d) Give all inflection points [x-coordinate only] of $f(x)$

Solution. There is only one inflection point, and it occurs at $x = 0$ [where it changes concavity]. □