1) [20 points] Consider the following permutations in S_7:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 4 & 5 & 6 \end{pmatrix} \qquad \text{and} \qquad \tau = (1,7,4)(1,3,5)(2,6) \quad \text{[note it's not disjoint!]}.$$

[**No need to show work for the items below!**]

(a) Write $\sigma \cdot \tau$ in the matrix representation [as σ was given].

Solution.

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 4 & 2 & 6 & 3 & 7 \end{pmatrix}.$$

□

(b) Write σ as a product of disjoint cycles.

Solution. $\sigma = (1,2,3)(4,7,6,5)$. □

(c) What is $|\sigma|$?

Solution. $|\sigma| = \operatorname{lcm}(3,4) = 12$.

□

(d) Write σ as a product of transpositions.

Solution. $\sigma = (1,3)(1,2)(4,5)(4,6)(4,7)$.

□

(e) Find ρ such that $\rho\tau\rho^{-1} = (2,7,5)(2,3,1)(4,6)$. If there is no such ρ, say so and justify.

Solution. We have $\rho(1) = 2$, $\rho(7) = 7$, $\rho(4) = 5$, $\rho(1) = 2$, $\rho(3) = 3$, $\rho(5) = 1$, $\rho(2) = 4$ and $\rho(6) = 6$. So:

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 5 & 1 & 6 & 7 \end{pmatrix}.$$

□

2) [20 points] Consider $D_6 = \{1, \rho, \rho^2, \dots, \rho^5, \phi, \rho\phi, \rho^2\phi, \dots, \rho^5\phi\}$ and its subgroup $H \stackrel{\text{def}}{=} \langle \rho^3, \phi \rangle$.

(a) Compute $(\rho^3\phi)^{231} \cdot (\rho^4\phi)^{-1} \cdot \rho^{601}$. [Your answer should be one of the listed elements above: ρ^i or $\rho^i\phi$, with $i \in \{0, \dots, 5\}$.]

Solution.

$$\begin{aligned}(\rho^3\phi)^{231} \cdot (\rho^4\phi)^{-1} \cdot \rho^{601} &= (\rho^3\phi) \cdot (\rho^4\phi) \cdot \rho \\ &= \rho^3(\phi\rho^4)\phi\rho = \rho^3(\rho^2\phi)\phi\rho = \rho^5\phi^2\rho = \rho^5\rho = \rho^6 = 1\end{aligned}$$

□

(b) List all the elements of H. [No need to justify or show work.]

Solution. $H = \{1, \rho^3, \phi, \rho^3\phi\}$. □

(c) Is $H \lhd D_6$? [Justify!]

Solution. No, as $\phi \in H$, but $\rho\phi\rho^{-1} = \rho^2\phi \notin H$.

□

3) [15 points] Show that $A_5 \not\cong D_{30}$. [Here, it suffices to give a *structural* property that one of the groups has, but the other does not.]

Proof. We have that A_5 is simple [i.e., the only normal subgroups are $\{1\}$ and the groups itself], but D_{30} is not. For instance, $|\langle \rho \rangle| = 30$, so it has index 2 in D_{30} and hence it is a proper normal subgroups different from $\{1\}$. [Or, $Z(D_{30}) = \{1, \rho^{15}\}$ is another example of a normal subgroups different from $\{1\}$.] $\square$

4) [20 points] Let $N \triangleleft G$ and $\phi \in \mathrm{Aut}(G)$. Show that $\phi(N) \triangleleft G$.

Proof. Let $y \in G$ and $m \in \phi(N)$. [We need to show that $ymy^{-1} \in \phi(N)$.] Since ϕ is a bijection [and hence onto], there is $x \in G$ such that $\phi(x) = y$. Also, by definition [of $\phi(N)$], there is $n \in N$ such that $\phi(n) = m$.

Then:

$$ymy^{-1} = \phi(x)\phi(n)\phi(x)^{-1} = \phi(x)\phi(n)\phi(x^{-1}) = \phi(xnx^{-1}).$$

Since $N \triangleleft G$, we have that $xnx^{-1} \in N$, and hence, $\phi(xnx^{-1}) = ymy^{-1} \in \phi(N)$.

$\square$

5) In this problem, we will prove that if $p \neq 2$ is a prime and G is a group with $|G| = 2p$, then G has a normal subgroup of order p. [It is also true for $p = 2$ and it can be done directly. But here we will assume that $p \neq 2$.] **You can use a previous item even if you haven't proved it!**

(a) [10 points] Assume that there is no subgroup of order p. Prove that G is then Abelian. [**Hint:** Use an old HW problem.]

Proof. If G has an element of order p, say x, then it has a subgroup of order p, namely $\langle x \rangle$. So, it cannot have such element. Therefore, by Lagrange, every element has order $2p$, 2 or 1.

If $|x| = 2p$, then $|x^2| = |x| / (2, |x|) = p/(2, p) = p$, which is a contradiction. So, no element has order $2p$, and hence every element has order 2 or 1.

Thus, for all $x \in G$, we have that $x^2 = 1$. As seen in a previous HW problem, this means that G is Abelian. □

(b) [10 points] Still assuming that there is no subgroup of order p, show that G has a subgroup, say N, of order 2. Since G is Abelian (by (a)), we have that $N \triangleleft G$. Derive a contradiction by looking at G/N.

Proof. Since every element has order 2 or 1 and only the identity has order one, we have that for any $x \in G \setminus \{1\}$, $N \stackrel{\text{def}}{=} \langle x \rangle$ has order 2.

Since $|N| = 2$, we have that $|G/N| = p$. So, an element $yN \in G/N \setminus \{1N\}$ has order p [as p is prime]. But, y must have order 2, as seen above. So, $(yN)^2 = y^2 N = N$, a contradiction since $|yN| = p > 2$. □

(c) [5 points] So, from the previous items, there is a subgroup of G, say H, of order p. Prove that $H \triangleleft G$.

Proof. Since $|G| = 2p$ and $|H| = p$, we have that $(G : H) = |G| / |H| = 2$, and hence it is normal. □

Proof. □