Midterm 2

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Walker), class notes and solutions to *our* HW problems *posted by me*. No other reference, including the Internet. Failing to follow these instructions will give you a zero in the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: noon on Wednesday (11/20). If you cannot bring it to class or to me, a scanned/typed copy by e-mail would be OK.

Definition: Let G be a finite Abelian group. Let $I(G)$ be the vector of *invariant factors* of G, i.e., $I(G) = (d_1, d_2, \ldots, d_k)$ if

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z},$$

where $d_{i+1} \mid d_i$ for $i \in \{1, \ldots, (k-1)\}$.

Let also $E(G)$ be the vector of *elementary divisors* [with the order from class, as explained below], i.e., $E(G) = (q_1, q_2, \ldots, q_l)$ if

$$G \cong \mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_l\mathbb{Z},$$

where q_i is a power of a prime [not necessarily distinct], say $q_i = p_i^{r_i}$, p_i prime, and if $i < j$ then either $p_i < p_j$ or we have both $p_i = p_j$ and $r_i \leq r_j$.
[Feel free to talk to me if this definition is not clear.]

1) Let

$$G_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/5^2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7^2\mathbb{Z}$$

$$G_2 = \mathbb{Z}/(2^3 \cdot 3^2 \cdot 5 \cdot 7^2)\mathbb{Z} \oplus \mathbb{Z}/(3^2 \cdot 5 \cdot 7)\mathbb{Z}$$

$$G_3 = \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/(3 \cdot 5^2 \cdot 7)\mathbb{Z} \oplus \mathbb{Z}/(2 \cdot 3^2)\mathbb{Z} \oplus \mathbb{Z}/(3 \cdot 7^2)\mathbb{Z}$$

(a) [6 points] Compute $E(G_i)$, $I(G_i)$ for $i = 1, 2, 3$.

(b) [7 points] Find all pairs $i, j \in \{1, 2, 3\}$ with $i < j$ such that $G_i \cong G_j$. [Justify!]

(c) [7 points] If $P_7 \in \mathrm{Syl}_7(G_2)$, then what is P_7 isomorphic to?

2) [20 points] The *exponent* of a group G is the smallest positive integer k [if exists] such that $g^k = 1$ for all $g \in G$. Prove that if G is a finite Abelian group, then its exponent [exists and] is d_1, where d_1 is the first entry of $I(G)$. [So, with the *additive* notation, we need to prove that d_1 is the smallest positive integer such that $d_1 \cdot g = 0$ for all $g \in G$.]

3) Let G be a finite Abelian group such that there is $H \leq G$ such that $H \neq \{0\}$ [using additive notation] and if $K \leq G$ and $K \neq \{0\}$, then $H \leq K$.

(a) [10 points] Prove that if such H exists, then G is cyclic of order power of a prime.

(b) [10 points] Give an example of a *non-Abelian* group of finite order for which such H does exist. [Just give me G and H. No need to justify.]

4) Let G be a group of order $3 \cdot 7 \cdot 11$.

(a) [10 points] Prove that G has normal subgroup, say H, of order 77.

(b) [10 points] Prove that if G does not have exactly 7 subgroups of order 3, then G is cyclic.

5) [20 points] Let p and q be distinct primes and let G be a group of order $|G| = p^2 q$. Prove that G has either a normal Sylow p-subgroup or a normal Sylow q-subgroup.
[**Hint:** This is about *sizes and counting*. To derive a contradiction, assume there is neither. Use the *Sylow Theorems* to get that $q > p$. Then, what are n_p and n_q?]