FINAL

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Walker), class notes and solutions to *our* HW problems *posted by me*. No other reference, including the Internet. Failing to follow these instructions will give you a zero in the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: 10am on Monday (12/09), which is the official time our final would end. If you cannot bring it to my office, a scanned/typed copy by e-mail would be OK.

1) Consider $D_6 = \{1, \rho, \rho^2, \ldots, \rho^5, \phi, \rho\phi, \rho^2\phi, \ldots, \rho^5\phi\}$. Show that $D_6/Z(D_6)$ is not Abelian. [**Hint:** Remember that $Z(D_6) = \langle \rho^3 \rangle$.]

2) Let G be a finite group with $|G| = n$ and $H \leq G$ with $(G : H) = k$.

(a) Prove that if n does not divide $k!$, then there is $N \leq H$ with $N \neq \{1\}$ and $N \triangleleft G$. [**Hint:** This is Problem 2.4.13.]

(b) Prove that if $k = p$, where p is the *least* prime dividing n, then $H \triangleleft G$. [Here n might be divisible by p more than once. In particular, n could be a power of p.] [**Hint:** This is related to the previous part! You can use it even if you could not do it.]

3) Let G be a group of order 190. Prove that is contains a normal subgroup of order 95. Moreover, prove that this subgroup is cyclic.

4) Let R be a ring with 1 [and $1 \neq 0$], M be an ideal of R and suppose that $R^\times = R \setminus M$. [Remember, $R \setminus M \stackrel{\text{def}}{=} \{x \in R \, : \, x \notin M\}$.] Prove that M is a maximal ideal and that there are no other maximal ideals besides M.

5) Let R be a *commutative* ring with 1 [and $1 \neq 0$] and I be an ideal of R. Let

$$\text{rad}(I) = \{x \in R \, : \, x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

(a) Prove that $\text{rad}(I)$ is an ideal of R containing I. [**Hint:** Newton's formula:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

works in *commutative* rings! [No need to prove it.]]

(b) Prove that if P is a prime ideal with $I \subseteq P$, then $\text{rad}(I) \subseteq P$.