1) [25 points] Let U be any set. Prove that there is a unique $A \in \mathscr{P}(U)$ such that for all $B \in \mathscr{P}(U)$, we have $A \cup B = B$.
[This very similar to HW problem on the last test, but we had $A \cap B = B$ instead of $A \cup B = B$. This current one was also done as an example in class and on a video.]

Proof. [Existence:] Let $A = \varnothing$. Then, for any $B \subseteq U$, we have that $A \cup B = \varnothing \cup B = B$.

[Uniqueness:] Suppose that A' has the same property, i.e., for all $B \in \mathscr{P}(U)$ we have $A' \cup B = B$. [We need to show that $A' = \varnothing$.] Then, on the one hand, since $\varnothing \in \mathscr{P}(U)$, we have that $A' \cap \varnothing = A'$ [by the assumed property of A'], but on the other hand $A' \cap \varnothing = \varnothing$ [as discussed in the existence part]. So, $A' = \varnothing$. □

2) [25 points] Prove that if x is an integer *not* divisible by 3, then $x^2 + 3x - 1$ *is* divisible by 3.
[**Hint:** If an integer n is *not* divisible by 3, then its remainder when divided by 3 is either 1 or 2. So, in other words, n is not divisible by 3 iff either $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.]

Proof. Following the hint, since x is not divisible by 3, we have that either $x = 3k + 1$ or $x = 3k + 2$, for some $k \in \mathbb{Z}$. We then divide the proof in cases:
Case 1: $x = 3k + 1$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} x^3 + 3x - 1 &= (3k+1)^2 + 3(3k+1) - 1 \\ &= (9k^2 + 6k + 1) + (9k + 3) - 1 \\ &= 9k^2 + 15k = 3(3k^2 + 5k). \end{aligned}$$

Since $k^2 + 5k \in \mathbb{Z}$ [as $k \in \mathbb{Z}$], we have that $3 \mid (x^3 + 3x - 1)$.

Case 2: $x = 3k + 2$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} x^3 + 3x - 1 &= (3k+2)^2 + 3(3k+2) - 1 \\ &= (9k^2 + 12k + 4) + (9k + 6) - 1 \\ &= 9k^2 + 21k + 9 = 3(3k^2 + 7k + 3). \end{aligned}$$

Since $k^2 + 7k + 3 \in \mathbb{Z}$ [as $k \in \mathbb{Z}$], we have that $3 \mid (x^3 + 3x - 1)$. □

3) [25 points] Suppose the $I \neq \varnothing$ is a set of indices and let $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ be indexed families of sets. Prove that

$$\bigcup_{i \in I}(A_i \cap B_i) \subseteq \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{i \in I} B_i\right).$$

[This is a HW problem, similar to the problem in the last exam.]

Proof. Let $x \in \bigcup_{i \in I}(A_i \cap B_i)$. So, for some $i_0 \in I$, we have that $x \in A_{i_0} \cap B_{i_0}$, i.e., $x \in A_{i_0}$ and $x \in B_{i_0}$.

The former means that $x \in \bigcup_{i \in I} A_i$, while the latter means that $x \in \bigcup_{i \in I} B_i$.

Hence, since both occur, we have that $x \in \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{i \in I} B_i\right)$. □

4) [25 points] Suppose the m and n are integers. Prove that if $m \cdot n$ is even, then either m or n is even. [This was an example done in class.]

Proof. Assume that mn is even and that m is odd. [We need to prove that n is even.] Suppose that n is odd. [Need then to derive a contradiction.] So, there are $k, l \in \mathbb{Z}$ such that $m = 2k+1$ and $n = 2l+1$. So, $mn = (2k+1)(2l+1) = 4kl + 2k + 2k + 1 = 2(kl + k + l) + 1$. Since $kl + k + l \in \mathbb{Z}$ [as $k, l \in \mathbb{Z}$], we have that mn is odd, a contradiction. Hence, n must be even. □