1) [20 points] Let a, b and c be integers such that $a \mid b$. Prove that $a \mid (b+c)$ if and only if $a \mid c$.

[**Remember:** If $x, y \in \mathbb{Z}$, then $x \mid y$ denotes "x divides y", which means that there is some $z \in \mathbb{Z}$ such that $y = x \cdot z$.]

Proof. Let $a, b, c \in Z$ and assume that $a \mid b$. Then, we have that $b = a \cdot k$ for some $k \in \mathbb{Z}$.

[$\rightarrow$] Assume that $a \mid (b+c)$. Then, $(b+c) = a \cdot l$ for some $l \in \mathbb{Z}$. Then, $c = (b+c) - b = a \cdot l - a \cdot k = a \cdot (l-k)$. Since $l - k \in \mathbb{Z}$ [since $k, l \in \mathbb{Z}$] we have that $a \mid (b+c)$.

[$\leftarrow$] Assume now that $a \mid c$. Then, $c = a \cdot l$ for some $l \in \mathbb{Z}$. Hence, $b+c = a \cdot k + a \cdot l = a \cdot (k+l)$. Since $k + l \in \mathbb{Z}$ [as $k, l \in \mathbb{Z}$], we have that $a \mid (b+c)$.

□

2) [20 points] Let U be any set. Prove that there is a unique $A \in \mathscr{P}(U)$ such that for all $B \in \mathscr{P}(U)$, we have $A \cap B = B$. [This was a HW problem.]

Proof. [Existence] Let $A = U$ and $B \in \mathscr{P}(U)$. Then, $B \subseteq U$ and hence $A \cap B = U \cap B = B$.

[Uniqueness] Suppose that for some $A' \in \mathscr{P}(U)$ we have that for all $B \in \mathscr{P}(U)$ we have $A' \cap B = B$. [We need to show $A' = U$.] In particular, we can take $B = U$ [as $U \subseteq U$ implies that $U \in \mathscr{P}(U)$] and then $A' \cap U = U$. On the other hand, since $A' \in \mathscr{P}(U)$, we have that $A' \subseteq U$, and thus $A' \cap U = A'$. Therefore, we have that $A' = U$.

□

3) [20 points] Prove that if x is an integer, then $x^2 + 3x + 1$ is odd. [This one is not exactly a HW problem, but a slight twist on one.]

Proof. Since $x \in \mathbb{Z}$, we have that x is either even or odd.

Case 1: Assume x is even. Then, $x = 2k$ for some $k \in k$. Hence,

$$x^2 + 3x + 1 = (2k)^2 + 6k + 1 = 4k^2 + 6k + 1 = 2 \cdot (2k^2 + 3k) + 1.$$

Since $2k^2 + 3k \in \mathbb{Z}$ [since $k \in \mathbb{Z}$], we have that $x^2 + 3x + 1$ is odd.

Case 2: Assume x is odd. Then, $x = 2k + 1$ for some $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} x^2 + 3k + 1 = (2k+1)^2 + 3(2k+1) + 1 = (4k^2 + 4k + 1) + (6k + 3) + 1 \\ = 4k^2 + 10k + 5 = 2 \cdot (2k^2 + 5k + 2) + 1. \end{aligned}$$

Since $2k^2 + 5k + 2 \in \mathbb{Z}$ [since $k \in \mathbb{Z}$], we have that x is odd. $\square$

4) [20 points] Suppose the $I \neq \varnothing$ is a set of indices and let $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ be indexed families of sets. Prove that

$$\bigcup_{i \in I} (A_i \setminus B_i) \subseteq \left(\bigcup_{i \in I} A_i \right) \setminus \left(\bigcap_{i \in I} B_i \right).$$

[This is from the book and was done in a video. You also did something similar in HW.]

Proof. Let $x \in \bigcup_{i \in I}(A_i \setminus B_i)$. Then, there is $i_0 \in I$ such that $x \in A_{i_0} \setminus B_{i_0}$, i.e., such that $x \in A_{i_0}$ and $x \notin B_{i_0}$.
Since $x \in A_{i_0}$, we have that $x \in \bigcup_{i \in I} A_i$.
Since $x \notin B_{i_0}$, we have that $x \notin \bigcap_{i \in I} B_i$.
Thus, $x \in \left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcap_{i \in I} B_i\right)$. $\square$

5) [20 points] Suppose $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are non-empty families of sets and for every $A \in \mathcal{F}$ and $B \in \mathcal{G}$, we have that $A \cup B \in \mathcal{H}$. Prove that $\bigcap \mathcal{H} \subseteq (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$. [This was a HW problem.]

Proof. Let $x \in \bigcap \mathcal{H}$. Then, for all $X \in \mathcal{H}$, we have that $x \in X$. Suppose that $x \notin \bigcap \mathcal{F}$, i.e., there is some $A \in \mathcal{F}$ such that $x \notin A$. [Need to show that $x \in \bigcap \mathcal{G}$.] Let $B \in \mathcal{G}$ [arbitrary]. Now, by assumption $A \cup B \in \mathcal{H}$ [with the $A \in \mathcal{F}$ and $B \in \mathcal{G}$ as above!]. Then, we have that $x \in A \cup B$. But, by assumption $x \notin A$, and hence $x \in B$. Since B was an arbitrary element of $\mathcal{G}$, we have that $x \in \bigcap \mathcal{G}$. $\square$