1) Prove that if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.
[This was a HW problem.]

Proof. We do the contrapositive: assume that neither A and C are disjoint nor B and D are disjoint. [Need to show that $A \times B$ and $C \times D$ are not disjoint.]
Since A and C are not disjoint, there is $a \in A \cap C$. Since B and D are not disjoint, there is $b \in B \cap D$. So, $(a, b) \in A \times B$ [as $a \in A$ and $b \in B$] and $(a, b) \in C \times D$ [as $a \in C$ and $b \in D$]. So, $(a, b) \in (A \times B) \cap (C \times D)$, and so $A \times B$ and $C \times D$ are not disjoint. □

2) [20 points] Let R be a relation from A to B and S be a relation from B to C. Prove that if $\mathrm{Ran}(R) \subseteq \mathrm{Dom}(S)$, then $\mathrm{Dom}(R) \subseteq \mathrm{Dom}(S \circ R)$.
[This was a *part of* a HW problem.]

Proof. Let $a \in \mathrm{Dom}(R)$. [Need $a \in \mathrm{Dom}(S \circ R)$.] Then, there is $b \in B$ such that $(a, b) \in R$. So, $b \in \mathrm{Ran}(R)$. Since $\mathrm{Ran}(R) \subseteq \mathrm{Dom}(S)$, we have that $b \in \mathrm{Dom}(S)$, i.e., there is $c \in C$ such that $(b, c) \in S$. Since $(a, b) \in R$ and $(b, c) \in S$, we have that $(a, c) \in S \circ R$. So, $a \in \mathrm{Dom}(S \circ R)$. □

3) Let A be the set of all people and R be the relation such that for $a, b \in A$, we have that aRb iff a and b have at least one common parent. Answer the questions below. [If the answer is affirmative, explain. If not, *give a counterexample!*]

(a) Is R reflexive?

Solution. Yes, since a person has the same parents as herself/himself. □

(b) Is R symmetric?

Solution. Yes, if a has a comment parent with b, then b has a comment parent with a [the same one]. □

(c) Is R transitive?

Solution. No, as person a might have the same father as person b [so aRb], but different mother, while b might have the same mother as c [so bRc], but a different father. In this case we do not have that aRc. □

(d) Is R antisymmetric?

Solution. No, if a and b are siblings [with $a \neq b$], then aRb and bRa, but $a \neq b$. □

4) [20 points] Let $A = \mathscr{P}(\mathbb{N})$,

$$B = \{\{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{1, 2, 3\}, \{5, 6, 7, 8, 9\}\}$$

and consider the ordering on A given by [the usual] "$\subseteq$". [**No need to justify your answers here!**]

(a) List all minimal elements of B. [If none, just say so.]

Solution. $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5, 6, 7, 8, 9\}$. □

(b) List all maximal elements of B. [If none, just say so.]

Solution. $\{4\}$, $\{1, 2, 3\}$ and $\{5, 6, 7, 8, 9\}$. □

(c) Give the greatest lower bound for B. [If none, just say so.]

Solution. It is $\bigcap B = \varnothing$. □

(d) Give the least upper bound for B. [If none, just say so.]

Solution. It is $\bigcup B = \{1, 2, 3, 4\}$. □

5) [20 points] Let R be an ordering relation on A and $B \subseteq A$. Prove that if there is $b \in B$ which is a lower bound for B, then it is also the smallest element of B *and* the greatest lower bound of B in A.
[So, there are two parts, but they are *really* simple and short!]

Proof. [As usual, since R is a partial order, I will denote xRy by $x \preccurlyeq y$.]
Let $x \in B$. Since b is a lower bound for B, we have that $b \preccurlyeq x$ [by definition of lower bound]. Then, since $b \in B$, we have that b is the least element of B [by definition of smallest element of B].
Now let $y \in A$ be a lower bound for b. [Need $y \preccurlyeq b$.] Since $b \in B$ and y is a lower bound for B, we have that $y \preccurlyeq b$. So, b is the greatest lower bound for B. □

Scratch: