1) [20 points] If

$$\begin{aligned} f &= 2 \cdot (x+2)^3 \cdot (x^2+2)^2 \cdot (x^2+3x+3) \cdot (x^3+x+1)^5, \\ g &= 3 \cdot (x+1)^5 \cdot (x^2+2) \cdot (x^2+3x+3)^3, \end{aligned}$$

are the factorizations of f an g into monic irreducible polynomials in $\mathbb{F}_5[x]$, then give the factorization of their GCD and LCM.

Solution. We have:

$$\begin{aligned} (f,g) &= (x^2+2)(x^2+3x+3), \\ [f,g] &= (x+1)^5(x+2)^3(x^2+2)^2(x^2+3x+3)^3(x^3+x+1)^5. \end{aligned}$$

□

2) [20 points] Let $f(x) = x^4 + x^2 + x + 1$ and $g(x) = x^3 + x^2 + x + 1$, both in $\mathbb{F}_2[x]$. Express their GCD as a linear combination of themselves.
[**Hint:** You should find that the GCD is $x + 1$.]

Solution. We have:

$$\begin{aligned} x^4 + x^2 + x + 1 &= (x^3 + x^2 + x + 1)(x + 1) + (x^2 + x) \\ x^3 + x^2 + x + 1 &= (x^2 + x)x + (x + 1) \\ x^2 + x &= (x + 1)x + 0. \end{aligned}$$

So, the GCD is $x + 1$. Then [remembering that in $\mathbb{F}_2$ we have $-1 = 1$]:

$$\begin{aligned} x + 1 &= (x^3 + x^2 + x + 1) + (x^2 + x)x \\ &= (x^3 + x^2 + x + 1) + [(x^4 + x^2 + x + 1) + (x^3 + x^2 + x + 1)(x + 1)]x \\ &= (x^4 + x^2 + x + 1)x + (x^3 + x^2 + x + 1)(x^2 + x + 1). \end{aligned}$$

□

3) [20 points] Let F be a field and $f, g, h \in F[x]$ with f and g relatively prime. Prove that if $f \mid h$ and $g \mid h$, then $(f \cdot g) \mid h$.
[**Hint:** If you could prove it in $\mathbb{Z}$ instead of $F[x]$, the same proof should work here. Also, this was a HW problem.]

Proof. By Bezout's Theorem [for polynomials], there are $r, s \in F[x]$ such that

$$rf + sg = 1.$$

So, multiplying by h, we get

$$rfh + sgh = h.$$

Now, since $f \mid h$ and $g \mid h$, we have that there are $f_1, g_1 \in F[x]$ such that $h = f_1 f = g_1 g$. Then, we get

$$h = rfh + sgh = rfg_1g + sgf_1f = fg(rg_1 + sf_1).$$

Since $rg_1 + sf_1 \in F[x]$, we have that $fg \mid h$. $\square$

4) [40 points] Decide if the polynomials below are irreducible or not in the corresponding polynomial ring. [Justify!]

(a) $f = x^2 - 3x + 5$ in $\mathbb{R}[x]$.

Solution. We have that $(-3)^2 - 4 \cdot 1 \cdot 5 = -11$, so it has no real roots. Since the degree is 2, we have that f is *irreducible.* $\square$

(b) $f = x^5 - x + 2$ in $\mathbb{C}[x]$.

Solution. Since it does not have degree one, by the *Fundamental Theorem of Algebra*, we have that f is *reducible.* $\square$

(c) $f = \dfrac{2}{3}x^3 + 4x^2 - 6x + \dfrac{4}{3}$ in $\mathbb{Q}[x]$.

Solution. We have $f = \dfrac{2}{3}(x^3 + 6x^2 - 9x + 2)$. So, let $f^\sharp = x^3 + 6x^2 - 9x + 2$ and then f is reducible iff $f^\sharp$ is. Now, since $f^\sharp$ has degree 3 is reducible iff it has a root. The possible rational roots are ± 1 and $\pm - 2$. Now, one can check that $f(1) = 0$, so $f^\sharp$ is reducible, and hence also f is *reducible*. $\square$

(d) $f = 3x^5 - 9x^4 + 6x^2 + 12x - 3335$ in $\mathbb{Q}[x]$.

Solution. We apply the "reversed Eisenstein Criterion" from Problem 3.91 from the textbook [and HW] with $p = 3$. Since $p \nmid 3335$ [as $3335 \equiv 2 \pmod 3$], but divides all coefficients, while $3^2 \nmid 3$. So, it's *irreducible*.

$\square$

(e) $f = x + 1000$ in $\mathbb{F}_{2017}[x]$.

Solution. It has degree one, so it is *irreducible*. $\square$

(f) $f = 1000x^3 - 999x^2 - 1001x + 20000$ in $\mathbb{Q}[x]$.

Solution. Reducing modulo 7, we get $\overline{f} = 6x^3 + 2x^2 + 1$. Now

$$\begin{aligned}
\overline{f}(0) &= 1 \neq 0, \\
\overline{f}(1) &= 2 \neq 0, \\
\overline{f}(2) &= 1 \neq 0, \\
\overline{f}(3) &= 6 \neq 0, \\
\overline{f}(4) &= 4 \neq 0, \\
\overline{f}(5) &= 3 \neq 0, \\
\overline{f}(6) &= 4 \neq 0,
\end{aligned}$$

and hence $\overline{f}$ has no roots. Since it has degree 3, it is irreducible. Hence, we have that f is *irreducible*.

[**Note:** You could use 11 instead of 7. The benefit is that it is easier to reduce modulo 11 [using Problem 1.80], but you have more possible roots to check.] $\square$

(g) $f = 3x^7 - 4x^6 + 18x^5 + 6x^4 + 2x^3 - 34x^2 + 100x - 30$ in $\mathbb{Q}[x]$.

Solution. It is *irreducible* by the Eisesntein Criterion for $p = 2$. $\square$

(h) $f = 2x^9 + 5x^7 + 3x^5 + x^4 + 6x^3 + 4x$ in $\mathbb{F}_7[x]$.

Solution. Clearly x is a factor:

$$f = x \cdot (2x^8 + 5x^6 + 3x^4 + x^3 + 6x^2 + 4).$$

So, f is *reducible*. $\square$