Midterm 1

Math 300 – Fall 2020

September 30th, 2020

1) Analyze the logical form of the following statements:

(a) Anyone who has bought a Rolls Royce must have a rich uncle. (Use the statement $B(x)$ for "x bought a Rolls Royce", $U(x, y)$ for "x is y's uncle", and $R(x)$ for "x is rich".)

Solution.

$$\forall x \left[B(x) \to (\exists y(U(y, x)) \land R(y))\right].$$

□

(b) "If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined." (Use $M(x)$ for "x has measles", $F(x, y)$ for "x and y are friends", $Q(x)$ for "x will have to be quarantined", and D for the *set* of everyone living in the dorm. Your quantifiers *cannot* be bound! If can only do it with bound quantifiers and you bound them correctly, you *do* get some partial credit.)

Solution.

$$(\exists x(x \in D \land M(x))) \to \left[\forall y \left([\exists z(z \in D \land F(y, z))] \to Q(y)\right)\right].$$

□

2) Negate the following statement and restate it as a positive statement.

$$\forall e \in \mathbb{R}_{>0} \left[\exists d \in \mathbb{R}_{>0} \left(\forall x(x < d \to x^2 < e)\right)\right].$$

In this problem you *can* have bound quantifiers.

Solution.

$$\begin{aligned}
\neg \left[\forall e \in \mathbb{R}_{>0} \left[\exists d \in \mathbb{R}_{>0} \left(\forall x(x < d \to x^2 < e)\right)\right]\right] \\
&\sim \exists e \in \mathbb{R}_{>0} \neg \left[\exists d \in \mathbb{R}_{>0} \left(\forall x(x < d \to x^2 < e)\right)\right] \\
&\sim \exists e \in \mathbb{R}_{>0} \left[\forall d \in \mathbb{R}_{>0} \neg \left(\forall x(x < d \to x^2 < e)\right)\right] \\
&\sim \exists e \in \mathbb{R}_{>0} \left[\forall d \in \mathbb{R}_{>0} \left(\exists x \neg(x < d \to x^2 < e)\right)\right] \\
&\sim \exists e \in \mathbb{R}_{>0} \left[\forall d \in \mathbb{R}_{>0} \left(\exists x(x < d \wedge x^2 \geq e)\right)\right].
\end{aligned}$$

□

3) Verify the equality

$$\bigcap_{i \in I}(A_i \setminus B_i) = \left(\bigcap_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right)$$

by showing (with logical symbols) that

$$x \in \bigcap_{i \in I}(A_i \setminus B_i) \sim x \in \left(\bigcap_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right).$$

[**Note:** This was a HW problem.]

Solution.

$$\begin{aligned}
x \in \bigcap_{i \in I}(A_i \setminus B_i) &\sim \forall i \in I(x \in A_i \setminus B_i) \\
&\sim \forall i \in I(x \in A_i \wedge x \notin B_i) \\
&\sim [\forall i \in I(x \in A_i)] \wedge [\forall i \in I \neg(x \in B_i)] \\
&\sim [\forall i \in I(x \in A_i)] \wedge \neg[\exists i \in I(x \in B_i)] \\
&\sim [x \in \bigcap_{i \in I} A_i] \wedge \neg[x \in \bigcup_{i \in I} B_i] \\
&\sim x \in \bigcap_{i \in I} A_i \setminus \bigcup_{i \in I} B_i.
\end{aligned}$$

□

4) Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$, then $x \in B$.

We use the contrapositive. Suppose that $x \notin B$. Since $x \in A$, we have that $x \in A \setminus B$. But since $A \setminus B \subseteq C \cap D$, this means that $x \in C \cap D$. In particular, we have that $x \in D$. Thus, if $x \notin B$, then $x \in D$, and therefore, if $x \notin D$, then $x \in B$. $\square$

5) Suppose that $y + x = 2y - x$, and x and y are not both zero. Prove that $y \neq 0$.
[**Note:** This was a HW problem.]

Proof. Suppose that $y = 0$. [We need to derive a contradiction.] Then, we have $0 + x = 2 \cdot 0 - x$, which gives $2x = 0$, or $x = 0$. Thus, we have that $x = 0$ and $y = 0$, which is a contradiction since we assumed that not both x and y were zero. $\square$