1) Suppose that R is a relation from A to B and S and is a relation from B to C. Prove that if $\text{Ran}(R) \subseteq \text{Dom}(S)$, then $\text{Dom}(R) \subseteq \text{Dom}(S \circ R)$.
[**Note:** This was (part of) a HW problem.]

Proof. Let $a \in \text{Dom}(R)$. Then, there is $b \in B$ such that $(a, b) \in R$. This also means that $b \in \text{Ran}(R)$, and since $\text{Ran}(R) \subseteq \text{Dom}(S)$, we have that $b \in \text{Dom}(S)$. Hence, there is $c \in C$ such that $(b, c) \in S$. Then, since $(a, b) \in R$ and $(b, c) \in S$, we have that $(a, c) \in S \circ R$, and thus $a \in \text{Dom}(S \circ R)$. Hence, $\text{Dom}(R) \subseteq \text{Dom}(S \circ R)$. $\square$

2) Let R_1 and R_2 be relations on a set A.

(a) Prove that if R_1 and R_2 are both reflexive, then so is $R_1 \cup R_2$.

Proof. Suppose that R_1 and R_2 are both reflexive and let $a \in A$. Then, since R_1 is reflexive, we have that $(a, a) \in R_1$, which means that $(a, a) \in R_1 \cup R_2$. $\square$

(b) Prove that if R_1 and R_2 are both symmetric, then so is $R_1 \cup R_2$.

Proof. Suppose that R_1 and R_2 are both symmetric and let $(a, b) \in R_1 \cup R_2$. Then either $(a, b) \in R_1$ or $(a, b) \in R_2$. In the former case, we have that $(b, a) \in R_1$, since R_1 is symmetric, and in the latter case we have that $(b, a) \in R_2$, since R_2 is symmetric. In either case we have that $(b, a) \in R_1 \cup R_2$. $\square$

3) Let R be a partial order on A, $B_1 \subseteq A$, $B_2 \subseteq A$, x_1 a least upper bound of B_1, and x_2 an upper bound of B_2. Prove that if $B_1 \subseteq B_2$, then $x_1 R x_2$.
[**Note:** This was a HW problem.]

Proof. We prove that x_2 is an upper bound of B_1: let $b \in B_1$. Since $B_1 \subseteq B_2$, we have that $b \in B_2$. Since x_2 is an upper bound of B_2, we have that $b R x_2$. Since $b \in B_1$ was arbitrary, we have that x_2 is an upper bound of B_1.
Now, since x_1 is the *least* upper bound of B_1, and x_2 is an upper bound of B_1, we have that $x_1 R x_2$. $\square$

4) Let m be a [fixed] positive integer and consider the relation on $\mathbb{Z}$ given by:

$$C_m = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid m \text{ divides } b - a\}.$$

Prove that C_m is an equivalence relation.
[Remember that m divides c iff $c = m \cdot k$ for some k in $\mathbb{Z}$.]

Proof. [Reflexive] Let $a \in \mathbb{Z}$. Then $a - a = 0 = 0 \cdot m$. Since $0 \in \mathbb{Z}$, we have that m divides $a - a$, and so $(a, a) \in C_m$.

[Symmetric] Suppose that $(a, b) \in C_m$. Then, $b - a = k \cdot m$ for some $k \in \mathbb{Z}$. Thus, $a - b = -(b - a) = -(k \cdot m) = (-k) \cdot m$. Since $-k \in \mathbb{Z}$, we have that m divides $a - b$, and so $(b, a) \in C_m$.

[Transitive] Suppose that $(a, b), (b, c) \in C_m$. Then $b - a = k \cdot m$ and $c - b = l \cdot m$ for some $k, l \in \mathbb{Z}$. Then,

$$c - a = (c - b) + (b - a) = l \cdot m + k \cdot m = (l + k) \cdot m.$$

Since $k + l \in \mathbb{Z}$, we have that m divides $c - a$, and so $(a, c) \in C_m$.

□

5) Let $f : A \to B$, $C \subseteq A$, and $g = f \cap (C \times B)$. Prove that $g : C \to B$ (i.e., the relation g is a function from C to B) and for all $c \in C$ we have that $g(c) = f(c)$.

Proof. Let $c \in C$. Then, since $C \subseteq A$, we have that $c \in A$. Then $(c, f(c)) \in f$. But, since $c \in C$ and $f(c) \in B$, we also have that $(c, f(c)) \in C \times B$. So, $(c, f(c)) \in f \cap (C \times B) = g$.

Suppose now that $(c, b) \in g$. Then, since $g \subseteq f$, we have $(c, b) \in f$, so $b = f(c)$. So, $(c, f(c))$ is the unique element of g with first coordinate c, and hence $g : C \to B$ and $g(c) = f(c)$. □