ALTERNATIVE SYMMETRIES AND SYSTEMS

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ABSTRACT. Traditionally, symmetries in music are based on geometry. More precisely they are symmetries of regular polygons (a dodecadon, most commonly), which, algebraically, are given by dihedral groups. In this paper we take an algebraic approach and discuss how to replace the traditional dihedral group with other groups that are still (algebraically) isomorphic to the original. In turn, this allows us to create new *musical systems* where not only the notion of symmetry is replaced, but the notion of interval as well. Two concrete examples are given by rewriting pieces by Webern and the band Muse using these new systems.

1. Introduction

Symmetries have been extensively used in music theory and composition. With the Pythagoreans, who already noted the symmetries in the natural scale, in the Renaissance, where the symmetric relations of reflection, augmentations, and diminution in modal counterpoint are explored, with Bach (Baroque), who also employed these relations in counterpoint but in the tonal system, and in late of the 19th century and especially in the 20th century, with the exploration of the regular divisions of the octave. (Messiaen's modes of limited transposition provide an excellent example of this kind of transpositive symmetry.)

This notion of symmetry in music is *geometric*, more precisely, they are based on symmetries of regular polygons, and thus come from *dihedral groups*, which give the symmetries of such objects. Our idea in this paper is to replace the classical dihedral groups with other subgroups of the symmetric group, and thus take an *algebraic* approach to symmetries, rather than geometric.

There are countless choices to replace the dihedral groups, some of which partially preserve geometric symmetries, some of which preserve only algebraic properties, and some which completely break with the usual structure, thus giving us a wide palette of choices.

These choices give new *musical systems* which can be used in compositions where the notion of symmetry is completely replaced by this algebraic view.

Here is a brief description of the contents of the remaining this paper. In Section 2 we review some of the mathematics necessary for the development of the theory. In Section 3 we discuss the a generalization of the *Webern matrix*, which can be used in twelve-tone compositions. In Section 4 we give a short analysis of two pieces: the first eight bars of the first movement of Webern's *Concerto for Nine Instruments, Op. 24*, and the first section on the song *Take a Bow* by Matthew Bellamy, with special attention give to their use of symmetries. In Section 5 we go back to mathematics and discuss possible alternatives to the dihedral group, while in Section 6 we use this alternatives to create alternative musical systems. In Section 8 we see how the new systems relate to permutations of pitch classes. In Section 7 we give some alternatives to Weben matrix in new systems, which are applied in Sections 9 to rewrite Webern's example from Section 4. Finally, in Section 10 we apply new systems to rewrite Muse's *Take a Bow* (analyzed in Section 4).

2. Mathematical Background for Symmetries

In this section we quickly review the mathematical background for the application of symmetries in music. Any introductory text in basic abstract algebra or group theory, e.g., [8] or [4], should provide the details for what follows. Also, many texts even establish the connection between group theory and music, e.g., [13].

We denote by S_n the *symmetric group* in n-elements, more precisely, the group of permutations of $\{0, 1\, \ldots, (n-1)\}$. Although it is more common in mathematics to see the elements of S_n as permutations on $\{1, 2, \ldots, n\}$, here it will be more convenient to take the former approach, as we identify $\{0, 1, 2, \ldots, 11\}$ with the *set of pitch space*, which is algebraically identified with $\mathbb{Z}/12\mathbb{Z}$ (integers modulo 12).

As mentioned above, symmetries in music are usually given by *dihedral groups*. Remember that the dihedral group D_n, for $n \geq 3$, is the group of symmetries of a regular n-gon. This group is generated by a rotation of $2\pi/n$ radians (or $360/n$ degrees) around the center, which we denote by ρ, and a reflection through a line bisecting one of its internal angles, which we denote by ϕ. Remember then that these elements satisfy $\rho^n = 1$, $\phi^2 = 1$, and $\phi\rho^k = \rho^{n-k}\phi$ for every k. Thus, the $2n$ elements of D_n can be represented as

$$D_n = \langle \rho, \phi \rangle = \{1, \rho, \rho^2, \ldots, \rho^{n-1}, \phi, \rho\phi, \rho^2\phi, \ldots, \rho^{n-1}\phi\}. \tag{2.1}$$

In music we see symmetries as permutations of pitches, or often, as permutations of *pitch classes*. Therefore, the geometric symmetries of the dihedral group needs to be identified with a subgroup of S_n. This is done by labeling the positions of the n vertices of a regular n-gon with numbers 0 through $n-1$. Assuming that the labels are ordered clockwise, the rotation ρ is also clockwise, and that the reflection ϕ is through the vertex labeled 0, we can represent ρ and ϕ, as elements of S_n (using *disjoint cycles representation*), as

$$\rho = (0\ \ 1\ \ 2\ \ 3\ \ \ldots\ \ n-1), \tag{2.2}$$

$$\phi = \begin{cases} (1\ \ n-1)(2\ \ n-2)\ldots((n-2)/2\ \ (n+2)/2), & \text{if } n \text{ is even,} \\ (1\ \ n-1)(2\ \ n-2)\ldots((n-1)/2\ \ (n+1)/2), & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

Therefore, seeing these as permutations of $\mathbb{Z}/n\mathbb{Z}$, we have that their actions on some $x \in \mathbb{Z}/n\mathbb{Z}$ are given by

$$\rho(x) = x + 1 \qquad \text{and} \qquad \phi(x) = -x. \tag{2.4}$$

Figures 2.1 and 2.2 illustrate the actions of ρ and ϕ, respectively, for $n = 5$. Note that in these, the numbers represent the *positions*, while the letters represent the physical vertices of the polygon.

From now on, when we refer to the dihedral group D_n, we will always mean the subgroup of S_n generated by ρ and ϕ as in Eqs. (2.2) and (2.3).

In concrete applications in music, the symmetries are either applied to the whole pitch class space, or one might only consider classes in a fixed scale, in which case we

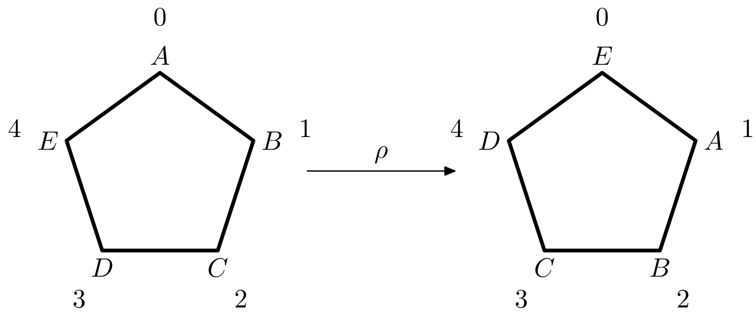

FIGURE 2.1. Action of ρ on the pentagon.

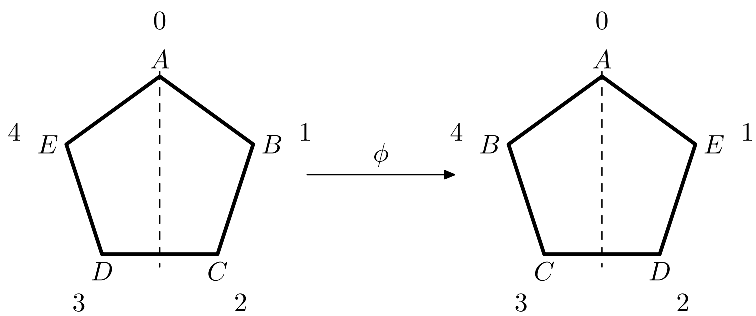

FIGURE 2.2. Action of ϕ on the pentagon.

can only regard seven of the pitch classes, and then consider the space to be $\mathbb{Z}/7\mathbb{Z}$ (instead of $\mathbb{Z}/12\mathbb{Z}$). Therefore, we work with either D_{12} (seen as a subgroup of S_{12}), the *modulo* 12 case, or D_7 (seen as a subgroup of S_7), the *modulo* 7 case.

We finish this section with some remarks about notation. We here mostly follow the common notation of mathematics, although the use of the letters ρ and ϕ are not universal. In music the map ρ^k is often denoted by $\mathbf{T}_k$, where the $\mathbf{T}$ stands for *transposition*. The map $\rho^k\phi$ is usually denoted by $\mathbf{T}_k\mathbf{I}$, where the $\mathbf{I}$ stands for *inversion*. (See [9, Chapter 2].) When focusing on music, rather than mathematics, we shall switch to this latter notation.

Finally, contrary to what is sometimes done in group theory, we denote the composition of functions in groups in the usual way we do it in mathematics in general. Therefore, if we write $\sigma \cdot \tau(k)$, we mean $\sigma(\tau(k))$, and *not* $\tau(\sigma(k))$.

3. The Webern Matrix

In Section 4 we will give some concrete examples of symmetries in music. One of them will be Webern's *Concerto for Nine Instruments*, which makes extensive use of symmetries. These symmetries are derived from *Webern's matrix*, which we describe in more detail below. In this section we shall discuss a general method to construct matrices similar to Webern's and describe their properties.

3.1. **Construction.** Consider a reordering of the twelve pitch classes, say

$$r_0 = (a_{0,0}, a_{0,1}, a_{0,2}, \ldots, a_{0,11}).$$

(So, $\{a_{0,0}, a_{1,1}, a_{0,2}, \ldots, a_{0,11}\} = \mathbb{Z}/12\mathbb{Z}$ as *sets*.) Here we use the double indices as usual for representation of matrices in mathematics: the first index represents the row while the second index represents the column.

Now, there is a single reflection $\rho^{i_0}\phi$ in D_{12} such that $\rho^{i_0}\phi(a_{0,0}) = a_{0,0}$. We then define

$$\begin{aligned} a_{i,0} &\stackrel{\text{def}}{=} \rho^{i_0}\phi(a_{0,i}), \quad \text{for } i \in \{1, 2, \ldots, 11\}, \\ c_0 &\stackrel{\text{def}}{=} (a_{0,0}, a_{1,0}, a_{2,0}, \ldots, a_{11,0}). \end{aligned}$$

So, each coordinate of c_0 is obtained by applying $\rho^{i_0}\phi$ to the corresponding coordinate of r_0. In this situation we will simply write $c_0 = \rho^{i_0}\phi(r_0)$.

Since both r_0 and c_0 start with $a_{0,0}$ by construction, we can start our 12×12 matrix by setting the first row as r_0 and the first column as c_0:

$$\begin{array}{ccccc} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,11} \\ a_{1,0} & & & & \\ a_{2,0} & & & & \\ \vdots & & & & \\ a_{11,0} & & & & \end{array}$$

Now, given some $k \in \{0, 1, \dots, 11\}$, there is a single rotation ρ^{j_k} such that $\rho^{j_k}(a_{0,0}) = a_{k,0}$. So, defining

$$\begin{aligned} a_{k,l} &\stackrel{\text{def}}{=} \rho^{j_k}(a_{0,l}) \quad \text{for } l \in \{1, 2, \dots, 11\}, \\ r_k &\stackrel{\text{def}}{=} (a_{k,0}, a_{k,1}, a_{k,2}, \dots, , a_{k,11}) = \rho^{j_k}(r_0), \end{aligned}$$

we can set the remaining rows of our matrix as r_k:

$$W(r_0) \stackrel{\text{def}}{=} \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,11} \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & a_{1,11} \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & a_{2,11} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{11,0} & a_{11,1} & a_{11,2} & \cdots & a_{11,11} \end{bmatrix}. \tag{3.1}$$

We might refer to the matrix $W(r_0)$ as the *Webern matrix of* r_0. (Note that it depends only on the choice of the first row r_0.)

Therefore, each row r_k in this matrix is obtained by applying some rotation ρ^{j_k} to the first row r_0, and the first column c_0 is obtained by applying a reflection $\rho^{i_0}\phi$ to the first row r_0.

Now, let

$$\begin{aligned} b_{i,j} &\stackrel{\text{def}}{=} a_{11-i,11-j} \quad \text{for } i, j \in \{0, 1, 2, \dots, 11\}, \\ r'_i &\stackrel{\text{def}}{=} (b_{i,0}, b_{i,1}, \dots, b_{i,11}). \end{aligned}$$

Then, for instance, we have that r'_0 be the last row of the matrix, but in *reversed order*, i.e.,

$$r'_0 = (a_{11,11}, a_{11,10}, a_{11,9}, \dots, a_{11,0}),$$

or, more generally, r'_i is r_{11-i} in reversed order, the so called the *retrograde* of r_{11-i}.

Similarly, let

$$c'_j = (b_{0,j}, b_{1,j}, \dots, b_{11,j}),$$

and hence c'_j is the column c_{11-j} in reversed order, i.e., its *retrograde*.

3.2. **Properties.** We now list the properties of the Webern matrix:

Proposition 3.1. *Let $W(r_0)$ be the Webern matrix of r_0 (given by Eq. (3.1)), as constructed in the previous section. We have:*

(1) *Each row r_k of $W(r_0)$ is obtained by applying some rotation of D_{12} to r_0.*
(2) *Each column c_l of $W(r_0)$ is obtained by applying some reflection of D_{12} to r_0.*
(3) *We have that $a_{k,k} = a_{0,0}$ for all $k \in \{1, 2, \ldots, 11\}$.*
(4) *Each retrograde row r'_k of $W(r_0)$ is obtained by applying some rotation of D_{12} to r'_0.*
(5) *Each retrograde column c'_l of $W(r_0)$ is obtained by applying some reflection of D_{12} to r'_0.*
(6) *If dividing the first row in d equal length parts we have that these parts are in the same set class, then every part of the division of any row and column in d equal parts is also in this same set class.*

Proof. Part 1 follows from the construction, as $r_k = \rho^{j_k}(r_0)$.

For Part 2, we start by observing that c_0 is obtained from r_0 by the reflection $\rho^{i_0}\phi$ by construction. So, we must now prove that the other columns are also obtained by reflections.

We have

$$c_l = (a_{0,l}, a_{1,l}, a_{2,l}, \ldots, a_{11,l}),$$

and we know that there is a (single) rotation, say ρ^{s_l} such that $\rho^{s_l}(a_{0,0}) = a_{0,l}$. Also, since $\rho^{j_k}(r_0) = r_k$ (by construction, or from the Part 1), we have that $\rho^{j_k}(a_{0,0}) = a_{k,0}$. Then, for any k, we have

$$\rho^{s_l}(a_{k,0}) = \rho^{s_l}(\rho^{j_k}(a_{0,0})) = \rho^{s_l+j_k}(a_{0,0}) = \rho^{j_k}(\rho^{s_l}(a_{0,0})) = \rho^{j_k}(a_{0,l}) = a_{k,l}.$$

But this means that $\rho^{s_l}(c_0) = c_l$, and thus c_l is obtained from c_0 by the rotation ρ^{s_l}. Hence, since $c_0 = \rho^{i_0}\phi(r_0)$, we have $c_l = \rho^{s_l}(c_0) = \rho^{s_l+i_0}\phi(r_0)$, proving Part 2.

For Part 3, we first note that, by construction, for all $k \in \{0, 1, \ldots, 11\}$ we have $\rho^{j_k}(a_{0,l}) = a_{k,l}$ for all $l \in \{0, 1, \ldots, 11\}$, and $\rho^{i_0}\phi(a_{0,k}) = a_{k,0}$ (which also means

$\rho^{i_0}\phi(a_{k,0}) = a_{0,k}$) for all k. Therefore,

$$\begin{aligned} a_{k,k} &= \rho^{j_k}(a_{0,k}) = \rho^{j_k}(\rho^{i_0}\phi(a_{k,0})) \\ &= \rho^{j_k+i_0}\phi(a_{k,0}) = \rho^{j_k+i_0}\phi(\rho^{j_k}(a_{0,0})) = \\ &= \rho^{j_k+i_0-j_k}\phi(a_{0,0}) = \rho^{i_0}\phi(a_{0,0}) \\ &= a_{0,0}. \end{aligned}$$

For Part 4, note that since $\rho^{j_k}(r_0) = r_k$, clearly we have $\rho^{j_k}(r'_{11}) = r'_{11-k}$, as $\rho^{j_k}(b_{11,j}) = \rho^{j_k}(a_{0,11-j}) = a_{k,11-j} = b_{11-k,j}$. Thus, we have that $\rho^{j_{11}}(r'_{11}) = r'_0$ and $\rho^{j_{11-k}}(r'_{11}) = r'_k$. Putting these together, we have that

$$\rho^{j_{11-k}-j_{11}}(r'_0) = \rho^{j_{11-k}}(r'_{11}) = r'_k,$$

proving Part 4.

For Part 5, remember that $\rho^{j_{11}}(r_0) = r_{11}$ and, by Part 2, given l there some i_l such that $\rho^{i_l}\phi(r_0) = c_l$, we have

$$c_{11-l} = \rho^{i_{11-l}}\phi(r_0) = \rho^{i_{11-l}}\phi\rho^{-j_{11}}(r_{11}) = \rho^{i_{11-l}+j_{11}}\phi(r_{11}).$$

The result follows from observing that c'_l and r'_0 are simply c_{11-l} and r_{11} in reversed order.

Finally, Part 6 is a triviality, since the rows and columns are obtained from elements of D_{12}. □

3.3. **The Original Webern Matrix.** We now describe Webern's original construction. We start with a set of three pitch classes: $\{0, 3, 11\}$. Then, taking $\sigma = \rho^7\phi, \rho^6, \rho\phi$, we have that $\{\sigma(0), \sigma(3), \sigma(11)\}$ becomes $\{4, 7, 8\}$, $\{5, 6, 9\}$, and $\{1, 2, 10\}$, respectively. (So, of course, these for sets are in the same *set class*, namely (014).) Note that these four sets give all the twelve pitch classes and therefore can be used in the general construction. We can then arrange the elements of each of these sets in an arbitrary way. Webern chose:

$$r_0 = (0, 11, 3, 4, 8, 7, 9, 5, 6, 1, 2, 10). \tag{3.2}$$

Note that this choice of r_0 satisfies the requirement of Part 6 of Proposition 3.1 in *two* ways: by dividing the first row in four trichords or in two hexachords. The former can be seen from the construction, while for the latter just note that both $\{0, 11, 2, 3, 8, 7\}$ and $\{9, 5, 6, 1, 2, 10\}$ belong to the set class (014589). Note also that these hexachords are fixed by exactly three rotations (i.e., transpositions) and three reflections (i.e., inversions) of D_{12}, namely $1 = \rho^0$, ρ^4, ρ^8, $\rho^3\phi$, $\rho^7\phi$, and $\rho^{11}\phi$.

Table 3.1 on the following page shows Webern's matrix $W(r_0)$, obtained by choosing the first row r_0 as in Eq. (3.2) above.

To make the symmetries explicit, we write on the left of the our representation of the Webern matrix the maps that give the rotation that take the first row to the corresponding row. For instance, since ρ^9 appears in the left of the third row, this row can be obtained by applying ρ^9 to the first row, i.e., $\rho^9(r_0) = r_2$. Similarly, the maps on the top of the matrix give the map that takes the first row to the corresponding column, the maps on right indicate the maps that takes r_0' to the corresponding retrograde row, and the ones on the bottom give the map that takes r_0' to the corresponding retrograde column.

Note that many other choices can be made to obtain a similar matrix. As we've seen, *any* row r_0 containing all pitch classes can be used to produce a matrix where the rows and columns are all related by symmetries.

3.4. **Notation in Music.** In music theory, the rows and columns of Webern's matrix are referred to as *series*, i.e., ordered sets. The rows themselves are called *prime orderings* and each one is denoted by the letter P (for prime) and with an index corresponding the first element of the series. So, for instance, we have that the third row of Webern's matrix (in Table 3.1) is P_9, since it starts with 9.

The columns of this matrix correspond to the so called *inversion orderings* and each one is denoted by the letter I and a subscript which is again the first element of the series.

We also have the *retrograde* and *retrograde-inversion* orderings, corresponding to rows and columns in reversed order. They are denoted by R and RI respectively, and the index now represents the *last* entry of the series. So, R_k is just P_k in reversed order, and RI_k is just I_k in reversed order.

	ϕ	$\rho^{11}\phi$	$\rho^3\phi$	$\rho^4\phi$	$\rho^8\phi$	$\rho^7\phi$	$\rho^9\phi$	$\rho^5\phi$	$\rho^6\phi$	$\rho\phi$	$\rho^2\phi$	$\rho^{10}\phi$	
1	0	11	3	4	8	7	9	5	6	1	2	10	ρ^{10}
ρ	1	0	4	5	9	8	10	6	7	2	3	11	ρ^{11}
ρ^9	9	8	0	1	5	4	6	2	3	10	11	7	ρ^7
ρ^8	8	7	11	0	4	3	5	1	2	9	10	6	ρ^6
ρ^4	4	3	7	8	0	11	1	9	10	5	6	2	ρ^2
ρ^5	5	4	8	9	1	0	2	10	11	6	7	3	ρ^3
ρ^3	3	2	6	7	11	10	0	8	9	4	5	1	ρ
ρ^7	7	6	10	11	3	2	4	0	1	8	9	5	ρ^5
ρ^6	6	5	9	10	2	1	3	11	0	7	8	4	ρ^4
ρ^{11}	11	10	2	3	7	6	8	4	5	0	1	9	ρ^9
ρ^{10}	10	9	1	2	6	5	7	3	4	11	0	8	ρ^8
ρ^2	2	1	5	6	10	9	11	7	8	3	4	0	1
	$\rho^2\phi$	$\rho\phi$	$\rho^5\phi$	$\rho^6\phi$	$\rho^{10}\phi$	$\rho^9\phi$	$\rho^{11}\phi$	$\rho^7\phi$	$\rho^8\phi$	$\rho^3\phi$	$\rho^4\phi$	ϕ	

Table 3.1. Webern's Matrix

4. Examples of Symmetries in Music

In this section we provide two examples of symmetries in composition. Since this section is dedicate to the applications in music, we shall use the notation most commonly used in this context. Therefore, the terms transposition and inversion will be used instead of rotation and reflection, which were used when dealing with the mathematical background. Similarly, we shall use $\mathbf{T}_n$ and $\mathbf{T}_n\mathbf{I}$ instead of ρ^n and $\rho^n\phi$ for the symmetry operations.

The examples we give here are the first eight bars of the first movement of Webern's *Concerto for Nine Instruments, Op. 24*, composed in 1934, and the first section on the song *Take a Bow* by Matthew Bellamy and recorded by the British band Muse in 2006.

Figure 4.1 on the next page shows the first eight bars from the first movement of Webern's *Concerto for Nine Instruments*. In this short segment, four series from

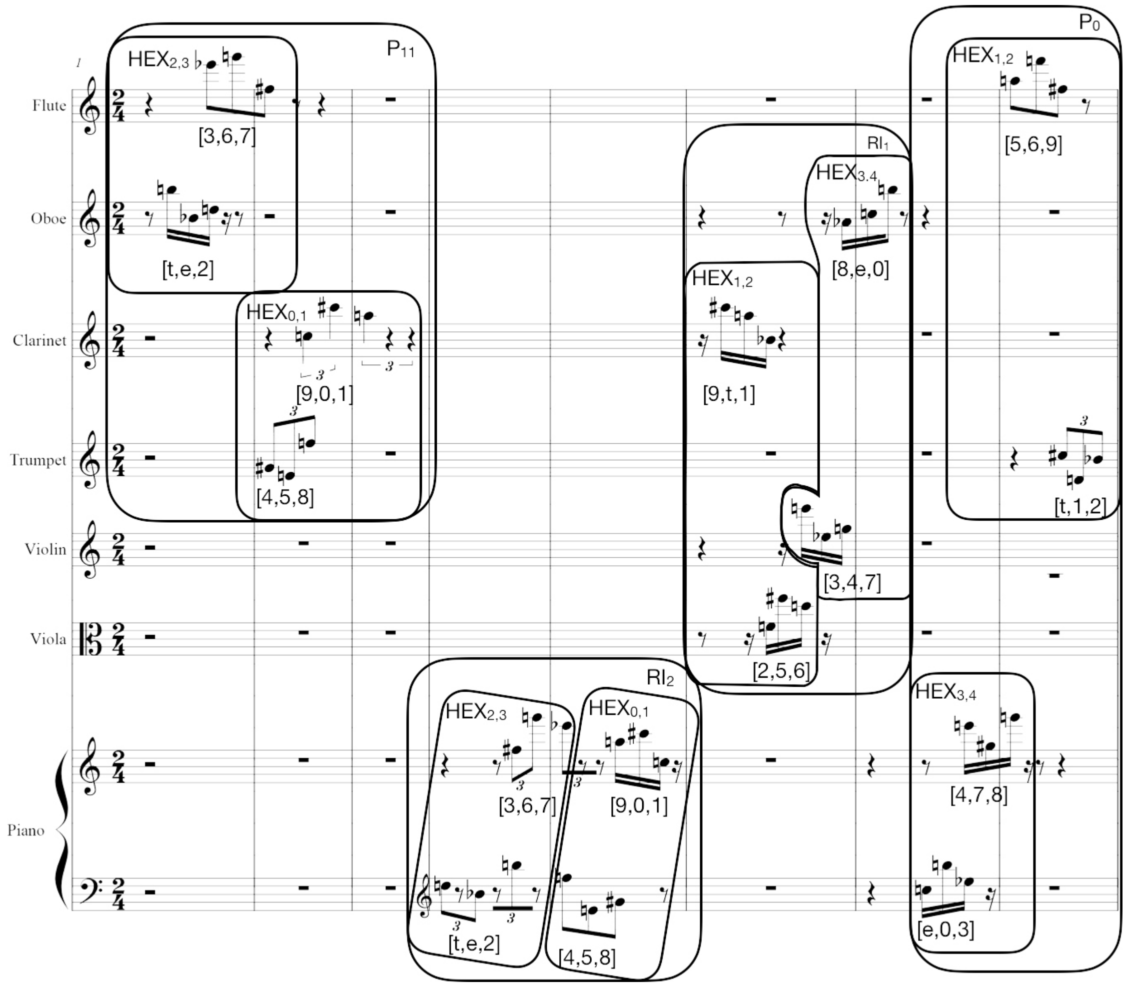

FIGURE 4.1. Webern's *Concerto for Nine Instruments, Op. 24, first movement, mm. 1-8.*

Webern's matrix (seen in Table 3.1) occur: P_{11} (mm. 1-3), RI_2 (mm. 4-5), RI_1 (mm. 6-7), and P_0 (mm. 7-8). Although clearly in any twelve-tone composition all series are related by symmetry, Figure 4.1 explicitly shows how in this work of Webern each series can be subdivided in smaller sets which are also related by symmetry. As observed in Section 3, each series is divided in two hexatonic collections of set class (014589) and each of these can be divided in two trichords of set class (014)[1]. Figure 4.2 on the following page shows the symmetry relations that connect these

[1]The division of the series of Webern's Concerto into two collections of hexatonics and four trichords of set class (014) was first observed by M. Babbitt in [7].

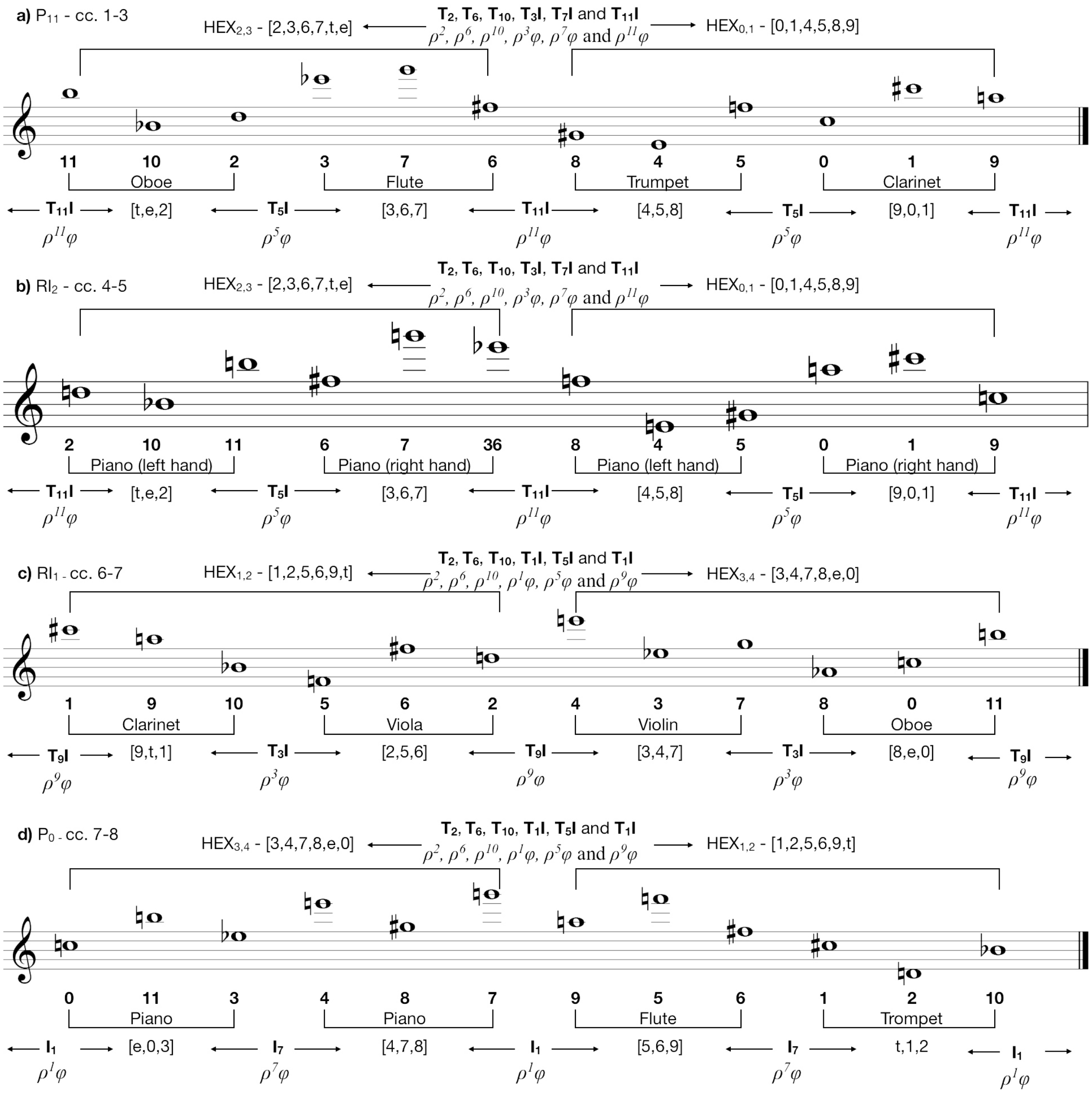

FIGURE 4.2. Symmetric operations relating the hexachords and trichords of the four series in the beginning of Webern's *Concerto*.

subseries, with the corresponding operations indicated below each series[2], showing that the series of Webern's *Concerto* also posses many *internal* symmetries.

The next example also explores the operations of transposition and inversion, but with a special focus on contextual inversions, which will be indicated with the labels

[2]Although the coefficients of the inversion operations change between the series, the second author's analysis [11, pgs. 27-30] showed that these operations are connected by the same axis of contextual inversion **J** and **D** in all series of Weber's *Concerto*.

of the transformations from Neo-Riemannian theory.[3] The first section of *Take a Bow* (mm. 1–65) by Matthew Bellamy is constructed entirely of major, minor, and augmented triads played by a synthesizer in ostinatos of arpeggios. The sequence of these arpeggios follows a chain of transformations $\langle \mathbf{PL'} \rangle^4$ for the members of the set class (037) of consonant triads, and thus all musical phrases with a length of eight bars of this section have its harmony polarized between two triads related by $\mathbf{L'}$. The connection between the phrases always occur between two triads related by $\mathbf{P}$. Therefore, the musical phrases repeat the same musical content under transposition. In Figure 4.3 on the next page one can observe the cycles formed by $\langle \mathbf{PL'} \rangle$ and the sequence of major and minor triads in this section of *Take a Bow*.

The augmented triads, set class (048), are introduced between the consonant triads that are related by $\mathbf{L'}$ to soften the voice leading and to make the connection between all triads happen by the displacement of single semitone.[5] Figure 4.4 on page 15 shows how the connections between these eighteen triads occur and how they relate with the operations of $\mathbf{T}_n$ and $\mathbf{T}_n\mathbf{I}$.

Figure 4.4 shows how the sequences of three triads from each musical phrase are connected by $\mathbf{T}_7$. It also shows that the triads that are related by transformations are operated by $\mathbf{T}_n\mathbf{I}$, where n is always even for the transformations $\mathbf{L'}$, and odd for the transformations $\mathbf{P}$. As the augmented triads are not in the same class as the consonant ones, they are not symmetrically related to the triads that come immediately before

[3]This Neo-Riemannian approach to the song *Take a Bow* was inspired by L. Bigo's video [1].

[4]In [6], R. Morris defines the transformations between the consonant triads according to how the intervals are preserved or moved in these operations: "$\mathbf{L}$ preserves ic 3, $\mathbf{P}$ preserves ic 5, and $\mathbf{R}$ preserves ic 4. To these are added what I call their obverse transforms, $\mathbf{L'}$, $\mathbf{P'}$, and $\mathbf{R'}$. In obverse operation, one note is held invariant while other two change. $\mathbf{L'}$ retains one note while the complementary ic 3 in the triad changes and therefore is related to L. P' and R' are similarly related to $\mathbf{P}$ and $\mathbf{R}$."

[5]This role that the set class (048) has of connecting two consonant triads that are two semitones apart in the voice leading, led the second author to refer to its members as *pivot sets* in [12]. Many graphs in Neo-Riemannian theory feature sets that play this role.

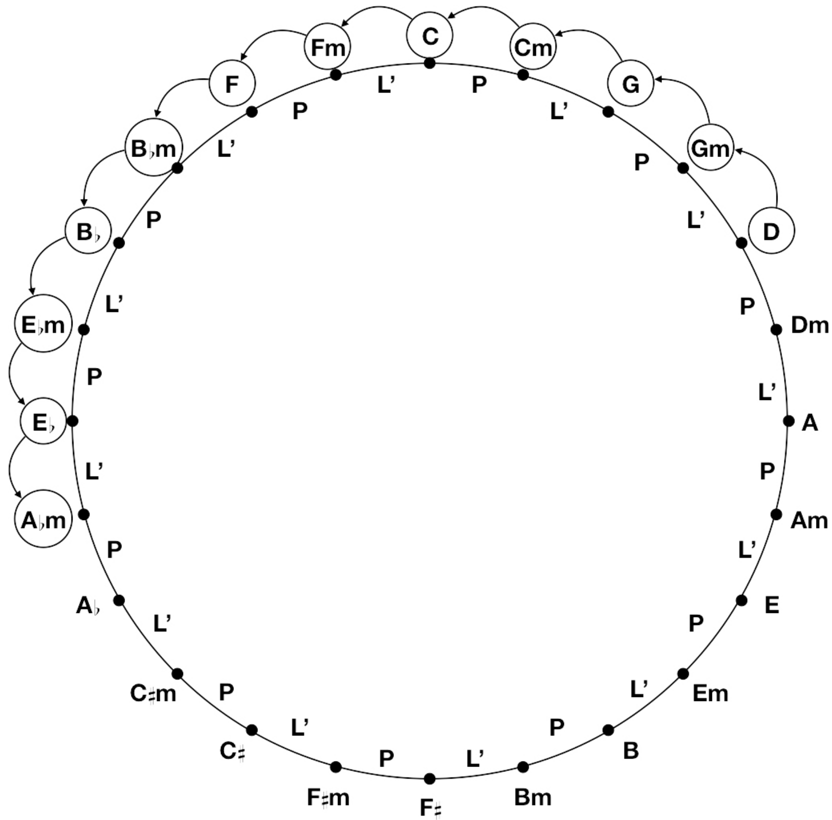

FIGURE 4.3. The cycle formed by the chain $\langle \mathbf{PL}' \rangle$ and the sequence of consonant triads in section A of *Take a Bow*.

or after them. As previously pointed out, their role in this sequence of triads is to soften the voice leading.

On the other hand, the set class (048) of the augmented triads is, just as the hexatonic collection, transpositionally and inversionally symmetric, i.e., the symmetric operations occur internally in each one of its members. This feature is common to almost all set classes that play the role of pivot. The parsimonious voice leading in the sequence of triads of section A of *Take a Bow* can be better visualized when traced in a graph created by Jack Douthett and Peter Steibach in [3], known as *Cube Dance*, shown in Figure 4.5 on page 16.

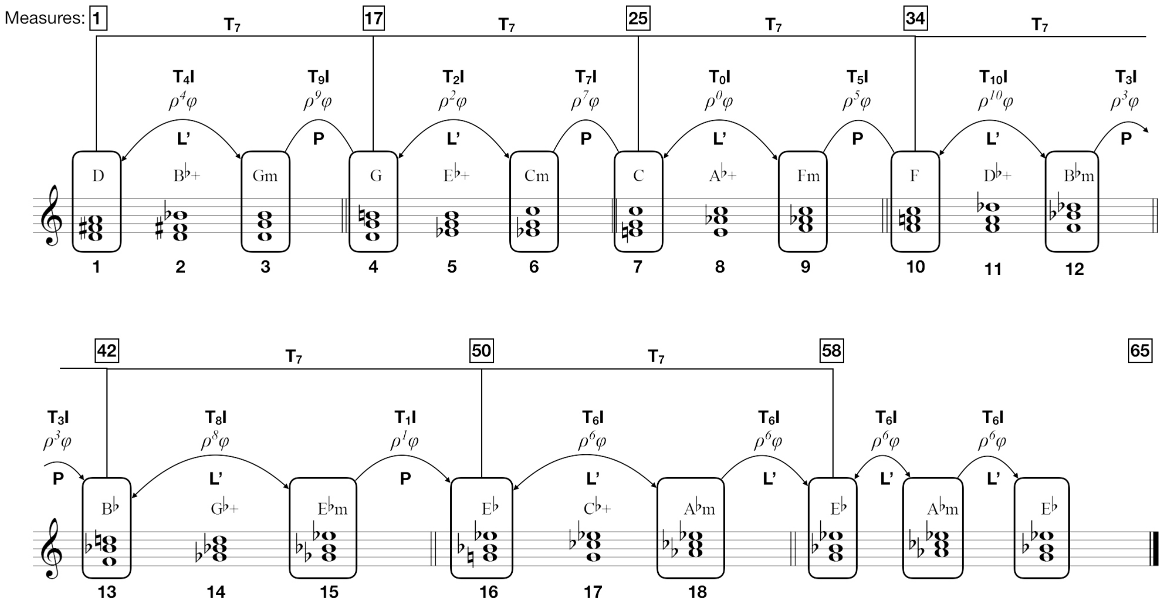

FIGURE 4.4. Sequence of all triads of section A of *Take a Bow* and the symmetry operations that relate them.

The path of the sequence of triads of section A of *Take a Bow* traced over the Cube Dance shows how it goes through the consecutive voice-leading zones in the counter-clockwise direction of the graph. Conceptualized by R. Cohn in [2], the voice-leading zones are formed by the triads that stay within the radius of the integers around the graph. This number represent the sum of the pitch classes of each triad, which is always in $\{1, 2, 4, 5, 7, 8, 10, 11\}$ for the consonant triads, and in $\{0, 3, 6, 9\}$ for the augmented ones. The voice-leading between triads connected by the edge in two consecutive zones is always parsimonious, and therefore another kind of symmetry, not measured by $\mathbf{T}_n$ or $\mathbf{T}_n\mathbf{I}$, occurs in this example, a symmetry in the voice-leading in which each connection is made by maintaining an interval of the two triads and displacing the remaining pitch by one semitone.

5. Alternative Dihedral Groups

We now return to mathematics. Remember the our main goal is to replace the dihedral groups D_{12} and D_7 by other subgroups of S_{12} and S_7 respectively. One first idea is to replace them by different subgroups that are still *isomorphic* to dihedral groups, and hence preserve all their algebraic properties. On the other hand, in the applications

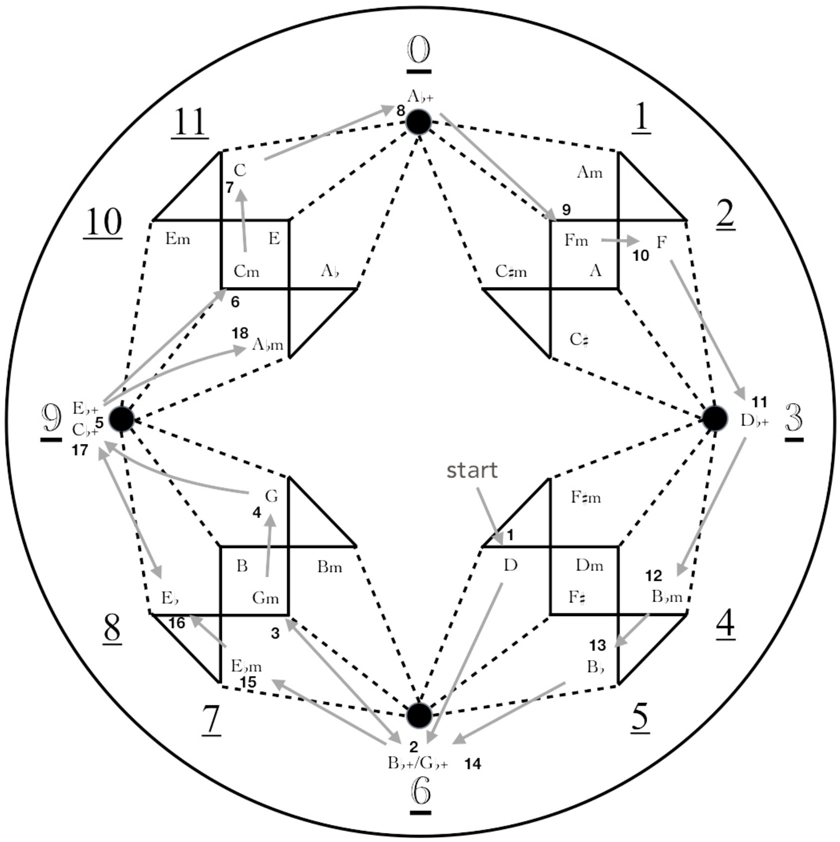

FIGURE 4.5. Sequence of all triads in section A of *Take a Bow* traced over the Cube Dance.

to music theory we discuss below, it is essential that these dihedral groups have a 12-cycle or a 7-cycle in the cases of D_{12} and D_7, respectively. Therefore, we will concentrate on these cases in this paper, leaving the discussion of the other cases to a future publication.

Note that a subgroup H of S_n is isomorphic to D_n if and only if there are elements $\hat{\rho}, \hat{\phi} \in H$ such that:

(1) $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$;
(2) $|\hat{\rho}| = n$;
(3) $\left|\hat{\phi}\right| = 2$;
(4) $\hat{\phi}\hat{\rho}^k = \hat{\rho}^{n-k}\hat{\phi}$ for all k.

In this case we have that

$$H = \{1, \hat{\rho}, \hat{\rho}^2, \dots, \hat{\rho}^{n-1}, \hat{\rho}\hat{\phi}, \hat{\rho}^2\hat{\phi}, \dots, \hat{\rho}^{n-1}\hat{\phi}\},$$

and an isomorphism $D_n \cong H$ can be given by the natural map $\Phi(\rho^k) = \hat{\rho}^k$, $\Phi(\rho^k\phi) = \hat{\rho}^k\hat{\phi}$, for $k \in \{0, 1, 2, \ldots, n-1\}$.

We now try to find *all* such subgroups of S_n for an arbitrary n, which we can then specialize to the cases of interest, namely $n = 12$ and $n = 7$. The first and easiest way to obtain such subgroups is to use *conjugation*: for each $\sigma \in S_n$, we have that $\sigma D_n \sigma^{-1} = \{\sigma\tau\sigma^{-1} \ : \ \tau \in D_n\}$ is as desired. For example, if we take, say,

$$\sigma = (0\ \ 9\ \ 2)(1\ \ 3\ \ 10\ \ 5\ \ 7\ \ 11\ \ 8\ \ 4\ \ 6),$$

when $n = 12$, then we get

$$\begin{aligned}\hat{\rho} &= \sigma\rho\sigma^{-1} = (9\ \ 3\ \ 0\ \ 10\ \ 6\ \ 7\ \ 1\ \ 11\ \ 4\ \ 2\ \ 5\ \ 8)\\ &= (0\ \ 10\ \ 6\ \ 7\ \ 1\ \ 11\ \ 4\ \ 2\ \ 5\ \ 8\ \ 9\ \ 3),\\ \hat{\phi} &= \sigma\phi\sigma^{-1} = (3\ \ 8)(0\ \ 5)(10\ \ 2)(6\ \ 4)(7\ \ 11)\\ &= (0\ \ 5)(2\ \ 10)(3\ \ 8)(4\ \ 6)(7\ \ 11).\end{aligned}$$

Notice, however, that not all different σ in S_n will give us different subgroups $\sigma D_n \sigma^{-1}$. For instance, if we take $\sigma = \phi$, then we would get $\hat{\rho} = \phi\rho\phi = \rho^{n-1}$ and $\hat{\phi} = \phi\phi\phi = \phi$. Thus, in this case, we have that $\sigma D_n \sigma^{-1} = D_n$. But our example above is clearly not D_{12} itself, as the $\hat{\rho}$ above has order 12 and the only elements of D_{12} of order 12 are

$$\rho = (0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11), \tag{5.1}$$

$$\rho^5 = (0\ \ 5\ \ 10\ \ 3\ \ 8\ \ 1\ \ 6\ \ 11\ \ 4\ \ 9\ \ 2\ \ 7), \tag{5.2}$$

$$\rho^7 = (0\ \ 7\ \ 2\ \ 9\ \ 4\ \ 11\ \ 6\ \ 1\ \ 8\ \ 3\ \ 10\ \ 5), \tag{5.3}$$

$$\rho^{11} = (0\ \ 11\ \ 10\ \ 9\ \ 8\ \ 7\ \ 6\ \ 5\ \ 4\ \ 3\ \ 2\ \ 1). \tag{5.4}$$

So, how many of the elements σ of S_n are such that $\sigma D_n \sigma^{-1} = D_n$? Observe that $\hat{\rho} = \sigma\rho\sigma^{-1}$ is an n-cycle, and thus has order n. Moreover, the only elements of order n in D_n are ρ^k, for $k \in \{1, 2, \ldots, (n-1)\}$ with $\gcd(n, k) = 1$. The number of such powers k is denoted by $\varphi(n)$, where φ is the so called *Euler φ-function.* (So, in general, we have that $\varphi(n)$ is simply the number of positive integers less than n and relatively prime to n.)

So, before we can count how many $\sigma \in S_n$ are such that $\sigma D_n \sigma^{-1} = D_n$, we count how many $\sigma \in S_n$ are such that $\sigma\rho\sigma^{-1} = \rho$. More generally, we have:

Proposition 5.1. *Let $\tau_1 = (a_1 \ \ a_2 \ \ \dots \ \ a_r)$ and $\tau_2 = (b_1 \ \ b_2 \ \ \dots \ \ b_r)$ be two r-cycles in S_n. Then, there are exactly $r((n-r)!)$ elements $\sigma \in S_n$ such that $\sigma\tau_1\sigma^{-1} = \tau_2$.*

Proof. We have that $\sigma\tau_1\sigma^{-1} = \tau_2$ if and only if

$$(\sigma(a_1) \ \ \sigma(a_2) \ \ \dots \ \ \sigma(a_r)) = (b_1 \ \ b_2 \ \ \dots \ \ b_r).$$

Since $(b_1 \ \ b_2 \ \ \dots \ \ b_r) = (b_2 \ \ b_3 \ \ \dots \ \ b_r \ \ b_1) = \cdots = (b_r \ \ b_1 \ \ \dots \ \ b_{r-1})$, we have r options for where σ can send a_1, namely, any of the r possible b_i's. But, after we choose where a_1 is sent, we know where all the a_i's are sent, as they are determined by the cycle structure. (For example, if $\sigma(a_1) = b_4$, then $\sigma(a_2) = b_5$, $\sigma(a_3) = b_6$, $\dots$, $\sigma(a_{r-3}) = b_r$, $\sigma(a_{r-2}) = b_1$, $\sigma(a_{r-1}) = b_2$, $\sigma(a_r) = b_3$.)

Now each element of the $(n-r)$ elements not moved by τ_1 can be sent to any of the $(n-r)$ elements not moved by τ_2. This gives $(n-r)!$ choices for each one of the r choices above. Hence, in total $r((n-r)!)$ possible σ's in S_n. $\square$

In our case, as $r = n$, for each ρ^k with $\gcd(k, n) = 1$, we have $n \cdot 0! = n$ elements $\sigma \in S_n$ such that $\sigma\rho\sigma^{-1} = \rho^k$. Since there $\varphi(n)$ possible exponents k, we have $n\varphi(n)$ elements $\sigma \in S_n$ such that $\sigma\rho\sigma^{-1}$ is another element of order n in D_n.

On the other hand, note that, in principle, not all of these possibilities might actually give $\sigma D_n \sigma^{-1} = D_n$, as we still need that $\sigma\phi\sigma^{-1} = \rho^i\phi$ for some $i \in \{0, 1, \dots, (n-1)\}$. But we claim that all these elements do give D_n:

Lemma 5.2. *There are exactly $\varphi(n) \cdot n$ elements $\sigma \in S_n$ such that $\sigma D_n \sigma^{-1} = D_n$.*

Proof. Let $\sigma \in S_n$ such that $\sigma D_n \sigma^{-1} = D_n$. As noted above, it is necessary then that $\sigma\rho\sigma^{-1} = \rho^k$ for some k with $\gcd(n, k) = 1$. As also seen above, there are exactly $\varphi(n) \cdot n$ such σ's, and so, to finish the proof, it suffices to show that if $\sigma\rho\sigma^{-1} = \rho^k$ for some k with $\gcd(n, k) = 1$, then $\sigma\phi\sigma^{-1} = \rho^i\phi$, for some $i \in \{0, 1, \dots, n-1\}$.

We first observe if we write $\rho^k = (0 \;\; a_1 \;\; a_2 \;\; \dots \;\; a_{n-1})$, then one of the n permutations σ that give $\sigma\rho\sigma^{-1} = \rho^k$ is given by the matrix representation

$$\sigma_k = \begin{pmatrix} 0 & 1 & 2 & \cdots & (n-1) \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}. \tag{5.5}$$

Moreover, seeing the entries $\{0, 1, \dots, n-1\}$ as elements of $\mathbb{Z}/n\mathbb{Z}$, one can see that, since $\rho = (0 \;\; 1 \;\; 2 \;\; \dots \;\; n-1)$, we have that $\sigma_k(i) = a_i = k \cdot i$. (See Eqs. (5.1) to (5.4) for the case when $n = 12$.)

Now, ϕ is the composition of *all* the two cycles $(a \quad -a)$ (again, with entries modulo n). Since $\gcd(n, k) = 1$, we have that the two cycles $(\sigma_k(a) \quad -\sigma_k(a)) = (ka \quad -ka)$ that make $\sigma_k \phi \sigma_k^{-1}$ are the exactly same two cycles that make ϕ, only perhaps in a different order. Hence, we have that $\sigma_k \phi \sigma_k^{-1} = \phi$.

Thus, with σ_k as above, we do have that $\sigma_k D_n \sigma_k^{-1} = D_n$.

Now, the remaining σ's such that $\sigma\rho\sigma^{-1} = \rho^k$ have matrix representations

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & (n-2) & (n-1) \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & \cdots & (n-2) & (n-1) \\ a_2 & a_3 & a_4 & \cdots & 0 & 1 \end{pmatrix}, \dots$$
$$\begin{pmatrix} 0 & 1 & 2 & \cdots & (n-2) & (n-1) \\ a_{n-2} & a_{n-1} & 0 & \cdots & a_{n-4} & a_{n-3} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & \cdots & (n-2) & (n-1) \\ a_{n-1} & 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \end{pmatrix}.$$

It's then easy to see that these are simply given by $\sigma_k \rho^i$, with σ_k as above (given by Eq. (5.5)) and $i \in \{1, 2, \dots, (n-1)\}$. But then:

$$\begin{aligned}
(\sigma_k \rho^i)\phi(\sigma_k \rho^i)^{-1} &= \sigma_k \rho^i \phi \rho^{-i} \sigma_k^{-1} && [\text{since } (ab)^{-1} = b^{-1}a^{-1}] \\
&= \sigma_k \rho^{2i} \phi \sigma_k^{-1} && [\text{since } \phi\rho^r = \rho^{-r}\phi] \\
&= \sigma_k \rho^{2i} \sigma_k^{-1} \sigma_k \phi \sigma_k^{-1} && [\text{since } \sigma_k^{-1}\sigma_k = 1] \\
&= (\sigma_k \rho \sigma_k^{-1})^{2i} (\sigma_k \phi \sigma_k^{-1}) && [\text{since } ab^r a^{-1} = (aba^{-1})^r] \\
&= (\rho^k)^{2i} \phi && [\text{since } \sigma_k \phi \sigma_k^{-1} = \phi \text{ and } \sigma_k \rho \sigma_k^{-1} = \rho^k] \\
&= \rho^{2ki} \phi,
\end{aligned}$$

where the exponent of ρ can be taken modulo n (since $|\rho| = n$). Thus, $(\sigma_k \rho^i) D_n (\sigma_k \rho^i)^{-1} = \langle \rho^k, \rho^{2ki}\phi \rangle$, which is equal to D_n.

Therefore, the $\varphi(n) \cdot n$ possible $\sigma \in S_n$ such that $\sigma\rho\sigma^{-1} = \rho^k$ with $\gcd(n,k) = 1$ are exactly all $\sigma \in S_n$ such that $\sigma D_n \sigma^{-1} = D_n$. $\square$

This lemma gives us the following result:

Theorem 5.3. *There are exactly $(n-1)!/\varphi(n)$ different subgroups of S_n that are conjugates of D_n (and therefore isomorphic to D_n), including D_n itself.*

Proof. We use the theory of group actions, which can be found in [8, Section 2.7].

We have that S_n *acts* on the set of its subgroups by conjugation. Consider then $A = \{\sigma D_n \sigma^{-1} \;:\; \sigma \in S_n\}$, i.e., A is the set of all conjugates of D_n, the so called *orbit* of the subgroup D_n by the group action. Lemma 5.2 basically states that the *stabilizer* of D_n by the action has order $\varphi(n)\cdot n$. Then, by [8, Theorem 2.142], we have that this orbit A has $|S_n|/(\varphi(n)\cdot n) = n!/(\varphi(n)\cdot n) = (n-1)!/\varphi(n)$ elements. $\square$

In particular, in our cases of interest, we have:

Corollary 5.4. *There are exactly $11!/\varphi(12) = 9{,}979{,}200$ different subgroups of S_{12} that are conjugates of D_{12} (one of them being D_{12} itself), and there are exactly $6!/\varphi(7) = 120$ different subgroups of S_7 that are conjugates of D_7 (one of them being D_7 itself).*

5.1. Dihedral Groups of Order $2n$ Containing an n-Cycle.

If almost 10 million subgroups isomorphic to D_{12} in S_{12} might not seem enough, one might ask if there are any others, as we only investigated those which arise from conjugation.

So, suppose $H \leq S_n$ with $H \cong D_n$. Then, $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$ with $|\hat{\rho}| = n$, $\left|\hat{\phi}\right| = 2$, and $\hat{\phi}\hat{\rho} = \hat{\rho}^{-1}\hat{\phi}$. (If $\Phi : D_{12} \to H$ is an isomorphism, then we can take $\hat{\rho} \stackrel{\text{def}}{=} \Phi(\rho)$ and $\hat{\phi} \stackrel{\text{def}}{=} \Phi(\phi)$.)

For $n = 12$, one can take $\hat{\rho} = (0\;\;1\;\;2)(3\;\;4\;\;5\;\;6)$ and $\hat{\phi} = (1\;\;2)(3\;\;6)(4\;\;5)$, and indeed we have that $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle \cong D_{12}$, but H is *not* a conjugate of D_{12}, as since the only elements of order 12 in D_{12} are 12-cycles, we know that no conjugation of D_{12}

can contain $\hat{\rho}$. And similarly any conjugate of H is a dihedral group isomorphic to, but not conjugate of D_{12}.

As it turns out, the case when $\hat{\rho}$ is an n-cycle (the only case we will consider here) consists of exactly the conjugates of D_n. In other words, we have:

Theorem 5.5. *There are exactly* $(n-1)!/\varphi(n)$ *subgroups of* S_n *that* contain an n-cycle $\hat{\rho}$ *and are isomorphic to* D_n*, which are exactly the conjugates of* D_n*.*

Proof. Suppose $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle \cong D_n$, with $\hat{\rho} = (a_0 \;\; a_1 \;\; \cdots \;\; a_{n-1})$ an n-cycle. We first want to show that there is some $\sigma \in S_n$ such that $H = \sigma D_n \sigma^{-1}$.

Since $\left|\hat{\phi}\right| = 2$, we have that $\hat{\phi}$ is either a 2-cycle or a product of *disjoint* 2-cycles. Moreover, since $\hat{\phi}\hat{\rho} = \hat{\rho}^{-1}\hat{\phi}$, we have $\hat{\phi}\hat{\rho}\hat{\phi}^{-1} = \hat{\rho}^{-1}$. Now,

$$
\begin{aligned}
\hat{\rho}^{-1} &= (a_0 \;\; a_{n-1} \;\; \cdots \;\; a_2 \;\; a_1) \\
&= (a_1 \;\; a_0 \;\; a_{n-1} \;\; \cdots \;\; a_3 \;\; a_2) \\
&\;\;\vdots \\
&= (a_k \;\; a_{k-1} \;\; \cdots \;\; a_{k+2} \;\; a_{k+1}) \\
&\;\;\vdots \\
&= (a_{n-1} \;\; a_{n-2} \;\; \cdots \;\; a_1 \;\; a_0)
\end{aligned}
$$

and

$$\hat{\phi}\hat{\rho}\hat{\phi}^{-1} = (\hat{\phi}(a_0) \;\; \hat{\phi}(a_1) \;\; \cdots \;\; \hat{\phi}(a_{11})) = \hat{\rho}^{-1}.$$

Thus, we must have that $\hat{\phi}(a_i) = a_{k-i}$ for some $k \in \{0, 1, 2, \ldots, (n-1)\}$ (independent of i), *where the indices are considered modulo* n. Note that then we have $\hat{\phi}(a_{k-i}) = a_{k-(k-i)} = a_i$. Hence, $\hat{\phi}$ is the product of all 2-cycles of the form $(a_i \;\; a_{k-i})$. (Note that if $a_i = a_{k-i}$, i.e., $i \equiv k - i \pmod{n}$, then $(a_i \;\; a_{k-i})$ is *not* a 2-cycle, and so it is not included in the representation of $\hat{\phi}$.)

Let then

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & (n-2) & (n-1) \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-2} \end{pmatrix}.$$

It is clear that $\sigma\rho\sigma^{-1} = \hat{\rho}$.

Now, remember that ϕ is the product of all 2-cycles of the form $(i \;\; -i) = (i \;\; 0-i)$. Then, it is easy to see that $\rho^k\phi$ is the product of all 2-cycles of the form $(i \;\; k-i)$.

We then have that $\sigma(\rho^k\phi)\sigma^{-1}$ is a product of 2-cycles of the form $(\sigma(i) \;\; \sigma(k-i)) = (a_i \;\; a_{k-i})$, and hence $\sigma(\rho^k\phi)\sigma^{-1} = \hat{\phi}$. Thus, $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle = \langle \sigma\rho\sigma^{-1}, \sigma(\rho^k\phi)\sigma^{-1} \rangle = \sigma\left(\langle \rho, \rho^k\phi \rangle\right)\sigma^{-1} = \sigma D_n \sigma^{-1}$.

Therefore, all subgroups H of S_n isomorphic to D_n and having an n-cycle are conjugates of D_n. Of course, the converse is trivial: all conjugates of D_n are isomorphic to D_n and contain an n-cycle, the conjugate of ρ.

Finally, Theorem 5.3 states that there are exactly $(n-1)!/\varphi(n)$ different conjugates of D_n, which then finishes the proof. $\square$

5.2. **Dihedral Groups Containing ρ.** Suppose we want to replace D_n with another subgroup of S_n that also contains ρ. Is that possible? In fact, Proposition 5.1 says that no, the only subgroup of S_n isomorphic to D_n and containing ρ is D_n itself. Indeed, if $H = \left\langle \rho, \hat{\phi} \right\rangle \cong D_n$, then $\hat{\phi}\rho(\hat{\phi})^{-1} = \rho^{-1}$. By Proposition 5.1, there are only $n((n-n)!) = n$ possible $\hat{\phi} \in S_n$. But we already know n possibilities, namely ϕ, $\rho\phi$, $\rho^2\phi$, ..., $\rho^{n-1}\phi$, and hence $\hat{\phi} = \rho^i\phi$, for some $i \in \{0, \dots, (n-1)\}$. But in this case we have $H = \left\langle \rho, \hat{\phi} \right\rangle = D_n$.

5.3. **Dihedral Groups Containing ϕ and an n-Cycle.** We can explicitly count the total number of dihedral subgroups of S_{12} that have ϕ and a 12-cycle.

Theorem 5.6. *There are exactly* 960 *distinct subgroups of* S_{12} *isomorphic to* D_{12} *(including* D_{12} *itself) that contain* ϕ *and a* 12*-cycle.*

Proof. Let H be such a subgroup. Then, $H = \langle \hat{\rho}, \phi \rangle$, where $\hat{\rho}$ is a 12-cycle. Let then:

$$\hat{\rho} = (0 \;\; a_1 \;\; a_2 \;\; \dots \;\; a_{11}).$$

Then, we must have $\phi\hat{\rho}\phi = \hat{\rho}^{-1}$, i.e.,

$$(0 \;\; \phi(a_1) \;\; \phi(a_2) \;\; \dots \;\; \phi(a_6) \;\; \dots \;\; \phi(a_{11})) = (0 \;\; a_{11} \;\; a_{10} \;\; \dots \;\; a_6 \;\; \dots \;\; a_1). \tag{5.6}$$

Looking at 6-th entries of these cycles, and since the first entries match, we must have that $\phi(a_6) = a_6$. But since $a_6 \neq 0$, the only possibility is $a_6 = 6$.

This leaves 10 possible choices for a_1: all integers from 1 to 11, except for 6. But, after that choice, since we have $\phi(a_1) = a_{11}$ by Eq. (5.6), there is only one choice for a_{11}.

This leaves 8 choices for a_2, and similarly, since $\phi(a_2) = a_{10}$ by Eq. (5.6), there is only one choice for a_{10}. Proceeding in a similar way, we obtain 6 choices for a_3, 4 choices for a_4, and 2 choices for a_5, while a_7, a_8, and a_9 have single choice. Hence, we have $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840$ possible $\hat{\rho}$'s.

Now, as before, for a given $\hat{\rho}$, we have that $\langle \hat{\rho}, \phi \rangle = \left\langle \hat{\hat{\rho}}, \phi \right\rangle$ if and only if $\hat{\hat{\rho}} = \hat{\rho}^k$ for some $k \in \{1, 5, 7, 11\}$. This means that there are $3840/4 = 960$ distinct dihedral groups $\langle \hat{\rho}, \phi \rangle$. $\square$

Note that most of those 960 subgroups will only contain ϕ and no other reflection from D_{12}. But there are some examples in which we *do* have another reflection. Before we show these examples, we start by proving that there is no other subgroup of S_{12}, besides D_{12} itself, containing ϕ and $\rho^2\phi$.

Theorem 5.7. *There is no subgroup $H = \langle \hat{\rho}, \phi \rangle$ of S_{12} such that $\hat{\rho}$ is a 12-cycle, $H \cong D_{12}$, $H \neq D_{12}$, and $\phi, \rho^2\phi \in H$.*

Proof. Suppose that $H = \langle \hat{\rho}, \phi \rangle$ with $\hat{\rho}$ a 12-cycle and $\phi, \rho^2\phi \in H$. Since $\phi, \rho^2\phi \in H$, we have that $\phi \cdot \phi\rho^2 = \rho^2 \in H$. Since $\rho^2 = (0\ \ 2\ \ 4\ \ 6\ \ 8\ \ 10)(1\ \ 3\ \ 5\ \ 7\ \ 9\ \ 11)$ and $H \cong D_{12}$, we must have that either $\hat{\rho}^2 = \rho^2$ or $\hat{\rho}^{-2} = \hat{\rho}^{10} = \rho^2$, as only $\hat{\rho}^2$ and $\hat{\rho}^{-2}$ give a product of two 6-cycles in H. Since $\langle \hat{\rho}, \phi \rangle = \langle \hat{\rho}^{-1}, \phi \rangle$, we may assume that $\hat{\rho}^2 = \rho^2$.

But, $\hat{\rho}^2 = \rho^2$ means that

$$\hat{\rho} = (0\ \ a\ \ 2\ \ a+2\ \ 4\ \ a+4\ \ 6\ \ a+6\ \ 8\ \ a+8\ \ 10\ \ a+10),$$

for some $a \in \{1, 3, 5, 7, 9, 11\}$ and where, as usual, we take these sums $a + 2i$ modulo 12. Now, since we must have $\phi\hat{\rho}\phi = \hat{\rho}^{-1}$, we have

$$\begin{aligned}(0 \;\; \phi(a) \;\; 10 \;\; \phi(a+2) \;\; 8 \;\; \phi(a+4) \;\; 6 \;\; \phi(a+6) \;\; 4 \;\; \phi(a+8) \;\; 2 \;\; \phi(a+10))& \\ = (0 \;\; a+10 \;\; 10 \;\; a+8 \;\; 8 \;\; a+6 \;\; 6 \;\; a+4 \;\; 4 \;\; a+2 \;\; 2 \;\; a)&.\end{aligned}$$

Comparing the odd-position coordinates, we have that $a + 2i = \phi(a + 10 - 2i) = 12 - (a + 10 - 2i)$, which means that $2a \equiv 2 \pmod{12}$. But this means that $a \equiv 1 \pmod 6$, so $a = 1$ or $a = 7$.

Taking $a = 1$ gives $\hat{\rho} = \rho$, while taking $a = 7$ gives $\hat{\rho} = \rho^7$. But we have $\langle \rho^7, \phi \rangle = \langle \rho, \phi \rangle = D_{12}$, which shows that $H = D_{12}$. □

On the other hand, we have:

Theorem 5.8. *There are exactly* 3 *subgroups* $H = \langle \hat{\rho}, \phi \rangle$ *of* S_{12} *such that:*

1. $\hat{\rho}$ *is a* 12*-cycle,*
2. $H \cong D_{12}$*,*
3. $H \neq D_{12}$*,*
4. $\{\phi, \rho^3\phi, \rho^6\phi, \rho^9\phi\} \subseteq H$*.*

(Note that in these cases we also have $\{\rho^3, \rho^6, \rho^9\} \subseteq H$*.) These subgroups are given by taking*

$$\hat{\rho} \stackrel{\text{def}}{=} (0 \;\; a \;\; 3-a \;\; 3 \;\; a+3 \;\; 6-a \;\; 6 \;\; a+6 \;\; 9-a \;\; 9 \;\; a+9 \;\; -a)$$

with a *equal to either* 4*,* 7*, or* 8*.*

Proof. The proof is similar. Suppose that H is such group. Since $\phi, \rho^3\phi \in H$, we have that $\rho^3 \in H$. Since $\rho^3 = (0 \;\; 3 \;\; 6 \;\; 9)(1 \;\; 4 \;\; 7 \;\; 10)(2 \;\; 5 \;\; 8 \;\; 11)$ and $H \cong D_{12}$, we must have that either $\hat{\rho}^3 = \rho^3$ or $\hat{\rho}^{-3} = \hat{\rho}^9 = \rho^3$, as only $\hat{\rho}^3$ and $\hat{\rho}^{-3}$ give a product of three 4-cycles in H. Since $\langle \hat{\rho}, \phi \rangle = \langle \hat{\rho}^{-1}, \phi \rangle$, we may assume that $\hat{\rho}^3 = \rho^3$.

But, $\hat{\rho}^3 = \rho^3$ means that

$$\hat{\rho} \stackrel{\text{def}}{=} (0 \;\; a \;\; b \;\; 3 \;\; a+3 \;\; b+3 \;\; 6 \;\; a+6 \;\; b+6 \;\; 9 \;\; a+9 \;\; b+9), \tag{5.7}$$

for some $a, b \in \{1, 2, 4, 5, 7, 8, 10, 11\}$ and where, again, we takes these sums $a + 3i$ and $b + 3i$ modulo 12. Now, since we must have $\phi\hat{\rho}\phi = \hat{\rho}^{-1}$, we have

$$\begin{aligned}(0\ \ \phi(a)\ \ \phi(b)\ \ 9\ \ \phi(a+3)\ \ \phi(b+3)\ \ 6\ \ \phi(a+6)\ \ \phi(b+6)\ \ 3\ \ \phi(a+9)\ \ \phi(b+9))\\ = (0\ \ b+9\ \ a+9\ \ 9\ \ b+6\ \ a+6\ \ 6\ \ b+3\ \ a+3\ \ 3\ \ b\ \ a).\end{aligned}$$

Comparing coordinates, we have that $a+3i = \phi(b+9-3i)$ and $b+3i = \phi(a+9-3i)$, which means that $a+b+9 \equiv 0 \pmod{12}$, i.e., $b \equiv 3-a \pmod{12}$. Therefore, $\hat{\rho}$ has the form given in the statement.

Thus, the possibilities for a are 1, 2, 4, 5, 7, 8, 10, 11. Of these, we have that $a = 1$ gives $\hat{\rho} = \rho$, while $a = 5$ gives $\hat{\rho} = \rho^5$, so both give $H = D_{12}$. By Eqs. (5.1) to (5.4), none of the others give D_{12} itself.

We now have to take into account which choices of a would give the same group. These choices have to produce two 12-cycles in the same group, so they must be in $\{\hat{\rho}, \hat{\rho}^5, \hat{\rho}^7, \hat{\rho}^{11}\}$. But, the form given above means that $\hat{\rho}^3(0) = 3$, and $(\hat{\rho}^k)^3(0) = 3$ (for $k \in \{1, 5, 7, 11\}$) only when $k = 5$. Therefore, two of the possible choices will give the same group.

We have that

$$\hat{\rho}^5 = \hat{\hat{\rho}} \stackrel{\text{def}}{=} (0\ \ 6-a\ \ a+9\ \ 3\ \ 9-a\ \ a\ \ 6\ \ \ -a\ \ a+3\ \ 9\ \ 3-a\ \ a+6).$$

So, with $a = 2$, we have that $\hat{\hat{\rho}}$ is the same as the $\hat{\rho}$ given by $a = 4$, and hence these two cases gives the same H.

Similarly, the cases $a = 7$ and $a = 11$, and $a = 8$ and $a = 10$, give the same groups. Thus, we have a total of 3 such H. □

Similarly, we have the following results:

Theorem 5.9. *There are exactly* 2 *subgroups* $H = \langle\hat{\rho}, \phi\rangle$ *of* S_{12} *such that:*

(1) $\hat{\rho}$ *is a* 12*-cycle,*
(2) $H \cong D_{12}$,

(3) $H \neq D_{12}$,
(4) $\{\phi, \rho^4\phi, \rho^8\phi\} \subseteq H$.

(Note that in these cases we also have $\{\rho^4, \rho^8\} \subseteq H$.) These subgroups are given by taking

$$\hat{\rho} \stackrel{\text{def}}{=} (0 \;\; a \;\; 2 \;\; 4-a \;\; 4 \;\; a+4 \;\; 6 \;\; 8-a \;\; 8 \;\; a+8 \;\; 10 \;\; -a),$$

with a equal to either 3 or 9.

Theorem 5.10. *There are exactly 15 subgroups $H = \langle \hat{\rho}, \phi \rangle$ of S_{12} such that:*

(1) $\hat{\rho}$ *is a* 12*-cycle,*
(2) $H \cong D_{12}$,
(3) $H \neq D_{12}$,
(4) $\{\phi, \rho^6\phi\} \subseteq H$.

(Note that in these cases we also have $\rho^6 \in H$.) These subgroups are given by taking

$$\hat{\rho} \stackrel{\text{def}}{=} (0 \;\; a \;\; b \;\; c \;\; 6-a \;\; 6-b \;\; 6 \;\; a+6 \;\; b+6 \;\; c+6 \;\; -b \;\; -a),$$

with $a = 1$, $b \in \{2, 4, 8, 10\}$ and $c \in \{3, 6\}$, except for $(a, b, c) = (1, 2, 3)$ which gives D_{12}, or with $a = 2$, $b \in \{1, 5, 7, 11\}$, and $c \in \{3, 6\}$.

Moreover, the cases with (a, b, c) equal to $(1, 8, 9)$, $(2, 1, 3)$, and $(2, 7, 9)$ are such that $\rho^3, \phi \in H$, which were already accounted for in Theorem 5.8.

Their proofs are follow quite similar ideas to the ones found on Theorem 5.8, and thus we will omit them here for the sake of brevity.

6. New Systems

Remember that for two pitch classes, say p_1 and p_2 in the pitch class space $\mathcal{P} \stackrel{\text{def}}{=} \mathbb{Z}/12\mathbb{Z}$, the *(ordered) pitch class interval* between p_1 and p_2 is simply $p_2 - p_1$ seen as an integer in $\{0, 1, 2, \ldots, 11\}$, i.e., the residue modulo 12 of $p_2 - p_1$.

We can change perspective a little, and observe that the pitch class interval between p_1 and p_2 is the integer k in $\{0, 1, 2, \ldots, 11\}$ such that $\rho^k(p_1) = p_2$. Thus, one can tie the idea of pitch class interval with the dihedral group D_{12}.

Hence, if one decides to replace the role of D_{12} by another subgroup of S_{12} isomorphic to D_{12} (i.e., still dihedral of order 24), say $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$, *and* $\hat{\rho}$ is also a 12-cycle, one can then redefine the notion of pitch class interval in terms of H: we define then the *pitch class interval* between p_1 and p_2 as the integer k in $\{0, 1, 2, \ldots, 11\}$ such that $\hat{\rho}^k(p_1) = p_2$. This clearly generalizes the original notion of pitch class interval.

Note that the requirement that $\hat{\rho}$ is 12-cycle is essential. For instance, if $\hat{\rho} = (0 \quad 1)(2 \quad 3 \quad 4 \quad 5)(6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11)$ and $\hat{\phi} = (3 \quad 5)(7 \quad 11)(8 \quad 10)$, then $\left\langle \hat{\rho}, \hat{\phi} \right\rangle \cong D_{12}$, but there is no power of $\hat{\rho}$ that would take 0 to 2, so we would not be able to define the interval between these pitch classes. (The authors have ideas for fix the definition of interval in these cases, but these will appear in a different article.)

It's important to note here that in this notion of interval depend on $\hat{\rho}$ itself, not simply on the group H. For instance, clearly, we have that $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle = \left\langle \hat{\rho}^5, \hat{\phi} \right\rangle$, but the choice between $\hat{\rho}$ and $\hat{\rho}^5$ affects how we measure intervals. Thus, even if we stick with $H = D_{12}$ itself, we can replace ρ by ρ^5, ρ^7, or ρ^{11} and obtain a new system. Of course, in that case, we do preserve all standard transpositions and inversions.

Remember further that the *unordered pitch class interval* or *interval class* between p_1 and p_2 is simply $\min\{p_1 - p_2, p_2 - p_1\}$, where again, these differences are considered modulo 12. But again, we can tie this notion with the dihedral group. Immediately, from the above, we see that if we let $k, l \in \{0, 1, 2, \ldots, 11\}$ such that $\rho^k(p_1) = p_2$ and $\rho^l(p_2) = p_1$, then the pitch class interval is $\min\{k, l\}$. (Of course, we have that $l = 12 - k$.) On the other hand, note that if $\rho^l(p_2) = p_1$ (and so $\rho^{-l}(p_1) = p_2$), then we have that $\rho^l(\phi(p_1)) = \phi(p_2)$, as

$$\rho^l(\phi(p_1)) = \phi(\rho^{-l}(p_1)) = \phi(p_2).$$

Therefore, we can redefine the interval class as $\min\{k, l\}$, where $k, l \in \{0, 1, 2, \ldots 11\}$, $\rho^k(p_1) = p_2$ and $\rho^l(\phi(p_1)) = p_2$. Then, clearly one can again generalize this for other dihedral groups $D = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$, with $\hat{\rho}$ a 12-cycle, by defining the *interval class* with respect to H as $\min\{k, l\}$, where $k, l \in \{0, 1, 2, \ldots 11\}$, $\hat{\rho}^k(p_1) = p_2$ and $\hat{\rho}^l(\hat{\phi}(p_1)) = p_2$.

Note that, since in H we have $\hat{\rho}\hat{\phi} = \hat{\phi}\hat{\rho}^{-1}$, we still can define this interval class as $\min\{k, l\}$, where $k, l \in \{0, 1, 2, \ldots 11\}$, $\hat{\rho}^k(p_1) = p_2$, and $\hat{\rho}^l(p_2) = p_1$, i.e., the interval class still does remain unordered (in the sense that the interval class between p_1 and p_2, and p_2 and p_1 are equal). Therefore, even though we can see how $\hat{\phi}$ relates to the notion of interval class, the actual choice of $\hat{\phi}$ does not affect it. It only depends on $\hat{\rho}$, and implicitly, on the fact that $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$ is dihedral. Therefore, replacing $\hat{\phi}$ by any other reflection $\hat{\rho}^i\hat{\phi}$, does not affect the interval class.

In particular, we shall adopt the following convention on the representation of dihedral groups as $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$: we shall always assume that $\hat{\phi}(0) = 0$, as if not, it can be replaced by some $\hat{\rho}^i\hat{\phi}$ that does fix 0, without changing H itself (and the notion of class interval). This convention helps $\hat{\phi}$ seem more familiar, as this is a property that the usual ϕ we also has.

This new concept of class interval allows us to create new *music systems*, where not only the notion of symmetry, originally given by $D_{12} = \langle \rho, \phi \rangle$, is replaced by some new dihedral group $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$, where $\hat{\rho}$ is a 12-cycle, but also the notion of ordered and unordered pitch class intervals is replaced these new ones associated to $\hat{\rho}$ (and $\hat{\phi}$).

Let's consider two examples to illustrate these new systems and how they affect interval classes. Let's consider the dihedral groups $H_1 \stackrel{\text{def}}{=} \langle \rho_1, \phi_1 \rangle$ and $H_2 \stackrel{\text{def}}{=} \langle \rho_2, \phi_2 \rangle$, where

$$\begin{aligned} \rho_1 &\stackrel{\text{def}}{=} (0\ \ 4\ \ 11\ \ 3\ \ 7\ \ 2\ \ 6\ \ 10\ \ 5\ \ 9\ \ 1\ \ 8) \\ \phi_1 &\stackrel{\text{def}}{=} \phi = (1\ \ 11)(2\ \ 10)(3\ \ 9)(4\ \ 8)(5\ \ 7), \end{aligned}$$

and

$$\begin{aligned} \rho_2 &\stackrel{\text{def}}{=} (0\ \ 10\ \ 4\ \ 1\ \ 8\ \ 5\ \ 11\ \ 2\ \ 3\ \ 9\ \ 6\ \ 7) \\ \phi_2 &\stackrel{\text{def}}{=} (1\ \ 9)(2\ \ 5)(3\ \ 8)(4\ \ 6)(7\ \ 10). \end{aligned}$$

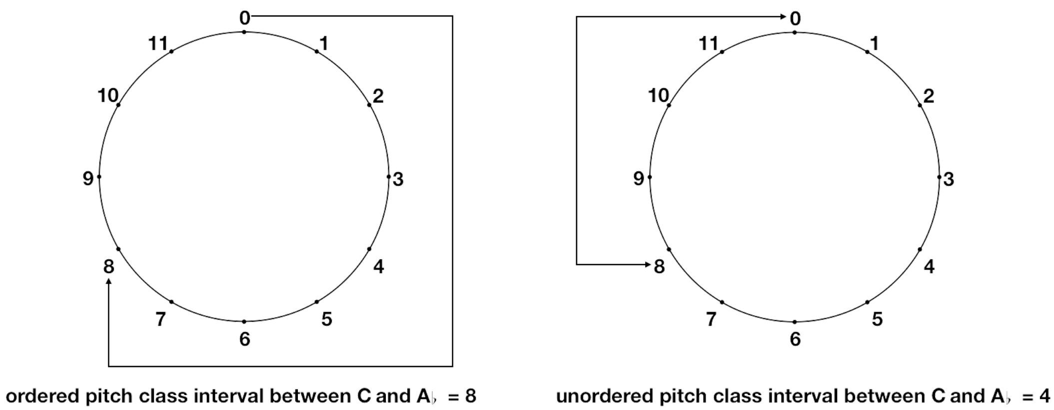

FIGURE 6.1. Ordered and unordered pitch class intervals between C and A♭ in the usual system.

Note that H_1 is one of the examples given in Theorem 5.8, and therefore we have $\rho^3, \phi \in H_1$, and hence it is close to the original D_{12}. On the other hand, the dihedral group H_2 has no element, other than the identity, in common with D_{12}. We shall use H_1 and H_2 as our main examples throughout.

Let's consider than intervals between the pitch classes of C (associated to 0) and A♭ (associated to 8). Then, the ordered pitch class interval between them is 8, since $\rho^8(0) = 8$. The unordered pitch class interval is 4, since $\rho^4(8) = 0$ (and $4 < 8$). These are often visualized in the "clock face", as seen in Figure 6.1. For the ordered class interval we move along the clock face clockwise. For unordered class interval, we take the smaller distance, either by going clockwise or counterclockwise.

When measuring pitch class intervals according to H_1 instead, we have that the ordered pitch class interval between C and A♭ is 11, since $\rho_1^{11}(0) = 8$, while the unordered pitch class interval is 1, since $\rho_1(8) = 0$ (and $1 < 11$). We can still visualize the interval classes in the clock face, but now instead of having the pitch classes move by a semitone, they are ordered according to $\rho_1 = (0\ \ 4\ \ 11\ \ 3\ \ 7\ \ 2\ \ 6\ \ 10\ \ 5\ \ 9\ \ 1\ \ 8)$. Therefore, after 0 comes 4, followed by 11, and so on. Figure 6.2 on the next page shows the clock face for the system given by H_1.

Similarly, with system given by H_2, we have that the ordered pitch class interval between C and A♭ is 4, since $\rho_2^4(0) = 8$. In this case, the unordered pitch class interval is also 4, since $\rho^8(8) = 0$ (and $8 > 4$). Figure 6.3 on the following page illustrates this case.

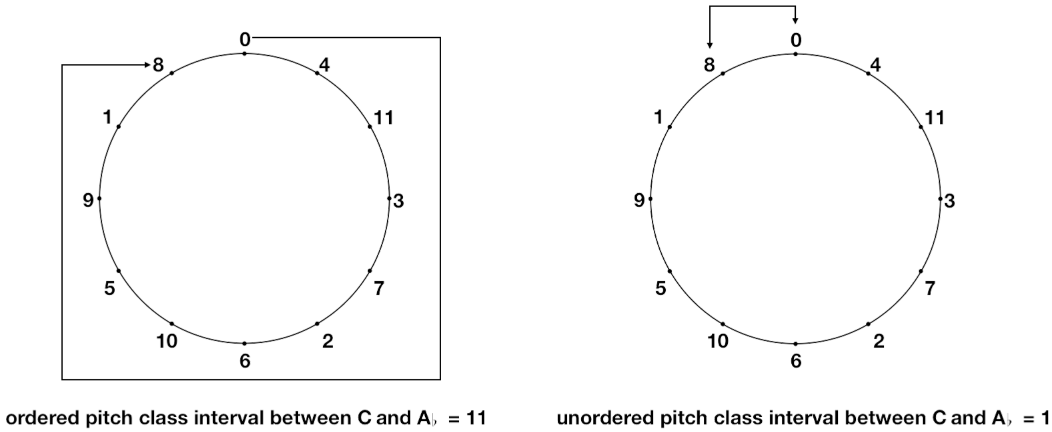

Figure 6.2. Ordered and unordered pitch class intervals between C and A♭ in the system given by H_1.

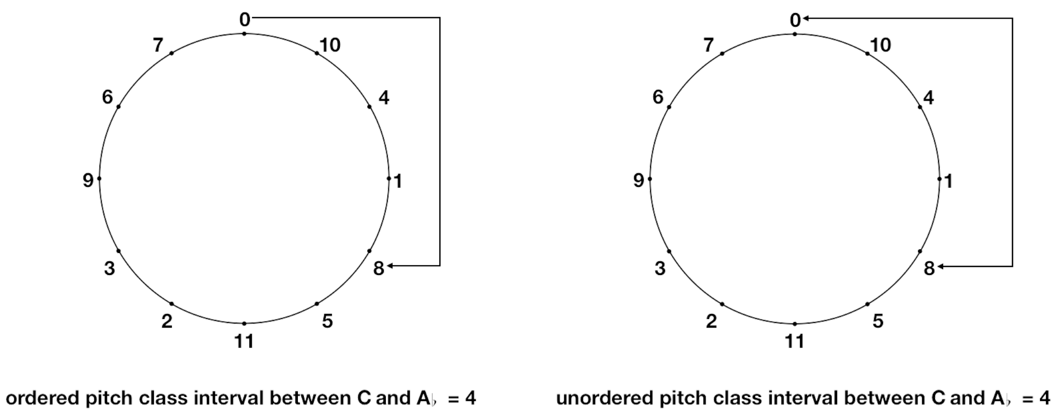

Figure 6.3. Ordered and unordered pitch class intervals between C and A♭ in the system given by H_2.

One can also use the new systems to measure pitch intervals (instead of pitch *class* intervals). In this case, the pitch interval measures the distance between two pitches in the pitch space. This can be visualized with straight line, with pitches going up by a semitone as we move to the right. In this case the ordered interval between C_4 and $A\flat_5$, represented by 0 and 20 respectively, is +20 and the ordered interval between $A\flat_5$ and C_4 is -20. The unordered interval in both cases is 20. (See Figure 6.4 on the next page.)

In the new systems we still assign 0 to C_4 and "unwrap" the clock face to the line accordingly. So, in the case of $\rho_1 = (0\ \ 4\ \ 11\ \ 3\ \ 7\ \ 2\ \ 6\ \ 10\ \ 5\ \ 9\ \ 1\ \ 8)$, we have 4 after 0, followed by 11, then 3, etc. After 8 we have 12, followed by 16, then 23,

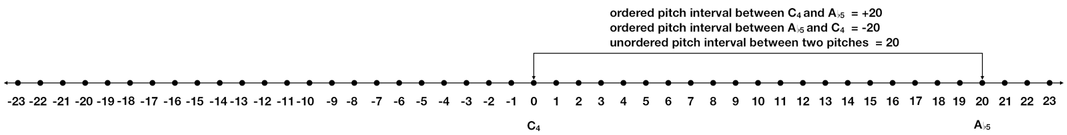

FIGURE 6.4. Interval between C_4 and $\mathrm{A}\flat_5$ in the usual system.

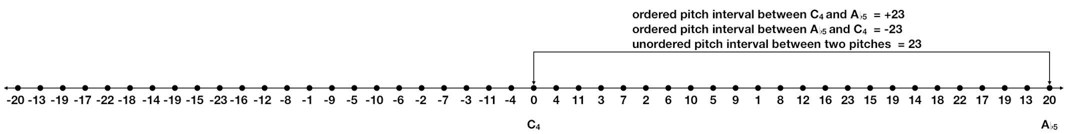

FIGURE 6.5. Intervals between C_4 and $\mathrm{A}\flat_5$ in the system given by H_1.

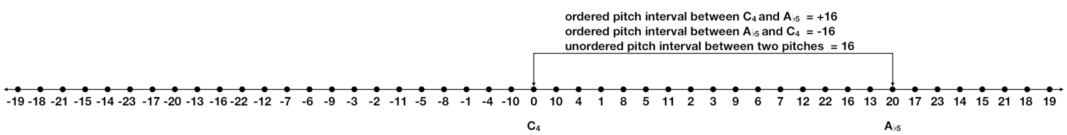

FIGURE 6.6. Intervals between C_4 and $\mathrm{A}\flat_5$ in the system given by H_2.

etc. Therefore, in this case the ordered interval between C_4 and $\mathrm{A}\flat_5$ is $+23$, and the ordered interval between $\mathrm{A}\flat_5$ and C_4 is -23. The unordered interval, in either order, is simply 23. Figure 6.5 illustrates this case.

Similarly, for the system given by H_2 we have that the ordered interval between C_4 and $\mathrm{A}\flat_5$ is $+16$, and the ordered interval between $\mathrm{A}\flat_5$ and C_4 is -16, and the unordered interval is simply 23, as illustrated in Figure 6.6.

As another illustration of these different systems, we shall consider their corresponding *Forte tables*, as described in [9, Appendix 1].

First, we observe that a *set class* of pitch class set can be thought as the orbit of the pitch class set by the dihedral group D_{12}. For example, if $S \subseteq \mathcal{P}$ is a set of pitch classes, say $S = \{p_1, p_2, \ldots, p_k\}$, then its set class is given by

$$(S) \stackrel{\text{def}}{=} \{\{\sigma(p) \ : \ p \in S\} \ : \ \sigma \in D_{12}\}.$$

It's usual to represent this class by its *prime form*, which makes a choice of representative that start with 0 and is most compressed to the right. (See [9, Chapter 2].)

We can use the same idea with a new system determined by a different dihedral group H containing a 12-cycle, and think of the class sets as orbits of the pitch class set now by H instead. Again, we use the prime form to represent elements, but note that the prime form is now determined by the corresponding way of measuring pitch class intervals.

For instance, the pitch class set $\{0, 1, 2\}$, in the standard equal temperament system (to which will refer from now simply as the *standard system*) has clearly prime form (012). On the other hand, in the system given by H_1, we have that this same set has prime form $(0et)$, since in this case, for example, the interval between 0 and 11 is only 2. Similarly, in the system given by H_2, the set has, *coincidentally* prime from (012) also. But note that in this last case the interval from 0 to 1 is 3 and not 1.

In fact, while in the standard system (012) has *interval vector* 210000, in the system given by H_2 the interval vector of (012) is 001110: the unordered pitch class interval from 0 to 1 is 3, from 1 to 2 is 4, and from 2 to 0 is 5.

In the case of the system given by H_1, the pitch set class $(0et)$ (of the pitch class set given by $\{0, 1, 2\}$) has interval vector 010020.

We shall now give examples of Forte tables given by these new systems. Note that traditionally the Forte table also includes identifying "names" each set class, but since they have no mathematical meaning and no analogue for different systems, we decided to omit them.

First, Table 6.1 on the facing page gives the Forte table for trichords and nonachords in the standard system. The corresponding Forte tables, again for trichords and nonachords, for the systems given by H_1 and H_2 are given by Tables 6.2 on the next page and 6.3 on page 34, respectively.

Finally we look at the concept of *sum class* (see [10]). Algebraically, the sum class of a pitch set or pitch class set is just the sum of the numerical values of the pitches or pitch classes in the set reduced modulo 12. On the other hand, as seen in cited reference, the notion comes from the idea of having a balance between ascending and descending motion. The sum class in the standard system captures this idea: if two set classes have the same sum class, then ascending and descending motions balance each other. This happens since the numerical value of the pitches or pitch classes

Trichords				**Nonachords**
(012)	210000	1, 1	876663	(012345678)
(013)	111000	1, 0	777663	(012345679)
(014)	101100	1, 0	767763	(012345689)
(015)	100110	1, 0	766773	(012345789)
(016)	100011	1, 0	766674	(012346789)
(024)	020100	1, 1	686763	(01234568t)
(025)	011010	1, 0	677673	(01234578t)
(026)	010101	1, 0	676764	(01234678t)
(027)	010020	1, 1	676683	(01235678t)
(036)	002001	1, 1	668664	(01234679t)
(037)	001110	1, 0	667773	(01235679t)
(048)	000300	3, 3	666963	(01245689t)

Table 6.1. Forte Table for Trichords and Nonachords.

Trichords				**Nonachords**
(036)	002001	1, 1	668664	(04e376t91)
(03t)	001110	1, 0	667773	(04e326t91)
(042)	100110	1, 0	766773	(04e372t59)
(043)	111000	1, 0	777663	(04e3726t9)
(046)	100011	1, 0	766674	(04e376t59)
(047)	101100	1, 0	767763	(04e372659)
(04e)	210000	1, 1	876663	(04e3726t5)
(075)	000300	3, 3	666963	(04e726591)
(0e2)	011010	1, 0	677673	(04e372t51)
(0e6)	010101	1, 0	676764	(04e376t51)
(0e7)	020100	1, 1	686763	(04e372651)
(0et)	010020	1, 1	676683	(04e326t51)

Table 6.2. Forte Table for Trichords and Nonachords for the system given by H_1.

can be interpreted as the intervals or interval class between the corresponding pitch or pitch class to the fixed reference of 0. This is not the case in new systems as, in general, the interval between 0 and a pitch or pitch class p is not p.

Trichords			Nonachords	
(012)	001110	1, 0	667773	$(0t415e296)$
$(01e)$	002001	1, 1	668664	$(0t418e296)$
(042)	010020	1, 1	676683	$(0t415e236)$
(045)	011010	1, 0	677673	$(0t4185236)$
(048)	020100	1, 1	686763	$(0t4185e36)$
$(04e)$	010101	1, 0	676764	$(0t418e236)$
(083)	000300	3, 3	666963	$(0t485e396)$
$(0t1)$	111000	1, 0	777663	$(0t4185e29)$
$(0t4)$	210000	1, 1	876663	$(0t4185e23)$
$(0t5)$	100110	1, 0	766773	$(0t4185239)$
$(0t8)$	101100	1, 0	767763	$(0t4185e39)$
$(0te)$	100011	1, 0	766674	$(0t418e239)$

Table 6.3. Forte Table for Trichords and Nonachords for the system given by H_2.

Therefore, in order to define a corresponding notion of the sum class in the new systems we need to take this interval notion of sum class, and not simply a numeric approach. Hence we define in any system the *sum class* of the pitch set $[a_1, a_2, \ldots, a_k]$ to be the reduction modulo 12 of the sum of the *intervals* between a_i and 0, and similarly, using interval classes, for pitch class sets. This then captures the idea that two pitch sets (or classes) have the same sum class if and only if the ascending and descending motions, measured with this notion of interval, are balanced.

For example, in the standard system the class sum of $[11, 0, 3]$ is $11 + 0 + 3 = 2$ (in $\mathbb{Z}/12\mathbb{Z}$). On the other hand, in the system given by H_1, we have that $[0, 11, 3]$ (the normal form of $\{0, 3, 11\}$ in this system) has sum class equal to $0 + 2 + 3 = 5$, as the interval between 0 and 0 is 0, between 0 and 11 is 2, and between 0 and 3 is 3. In the system given by H_2, the class sum of $[11, 3, 0]$ is $6 + 8 + 0 = 2$.

We close this section with a final observation about notation. We shall use the notation $\mathbf{T}_n$ and $\mathbf{T}_n\mathbf{I}$ for the corresponding transposition and inversion in every system. While in the standard system these correspond to ρ^n and $\rho^n\phi$, respectively, in the system given by H_1, for instance, they correspond to ρ_1^n and $\rho_1^n\phi_1$ instead.

7. New Webern Matrices

We can also use the other systems, i.e., other dihedral groups, to create Webern matrices.

We start with an example for H_1. We first need to find the first row. Choosing the set class (042) (in this system), for example, we observe that

$$\begin{aligned} \rho_1\phi_1(\{0,4,2\}) &= \{4,0,5\}, \\ \rho_1^7\phi_1(\{0,4,2\}) &= \{10,6,11\}, \\ \rho_1^4(\{0,4,2\}) &= \{7,2,9\}, \\ \rho_1^{10}(\{0,4,2\}) &= \{1,8,3\}, \end{aligned}$$

and thus we can take, for instance, the first row to be

$$r_0 = (0,5,4,11,6,10,7,9,2,3,1,8),$$

which yields the Webern matrix in Table 7.1 on the next page.

Note that the first and second hexachords, $[0,4,11,6,10,5]$ and $[3,7,2,9,1,8]$, have two inner symmetries by rotations (the identity and ρ_1^6) and two inner symmetries by reflection ($\rho_1^2\phi$ and $\rho_1^8\phi_1$), and the first can be mapped into the second by two rotations (ρ_1^3 and ρ_1^9) and two reflections ($\rho_1^5\phi_1$ and $\rho_1^{11}\phi_1$). Thus, as in Webern's original series, this new example also posses many internal symmetries, not only for trichords, but also for hexachords. (Although here the hexachords have only two, instead of three, symmetries by transposition and reflection, there are many choices of the first row/series in this system that would also yield three of each instead.)

Although this system is fairly close to the standard one, as $\rho_1^i = \rho^i$ for $i = 3,6,9$, and $\phi_1 = \phi$, and thus have three of the eleven non-trivial transpositions and four of the twelve inversions, using other dihedral groups can yield widely different systems. The example of H_2 illustrates this point.

	ϕ	$\hat\rho^8\phi$	$\hat\rho\phi$	$\hat\rho^2\phi$	$\hat\rho^6\phi$	$\hat\rho^7\phi$	$\hat\rho^4\phi$	$\hat\rho^9\phi$	$\hat\rho^5\phi$	$\hat\rho^3\phi$	$\hat\rho^{10}\phi$	$\hat\rho^{11}\phi$	
1	0	5	4	11	6	10	7	9	2	3	1	8	$\hat\rho^{11}$
$\hat\rho^4$	7	0	2	6	1	8	5	4	9	10	11	3	$\hat\rho^3$
$\hat\rho^{11}$	8	10	0	4	2	6	3	5	7	11	9	1	$\hat\rho^{10}$
$\hat\rho^{10}$	1	6	8	0	7	2	11	10	3	4	5	9	$\hat\rho^9$
$\hat\rho^6$	6	11	10	5	0	4	1	3	8	9	7	2	$\hat\rho^5$
$\hat\rho^5$	2	4	6	10	8	0	9	11	1	5	3	7	$\hat\rho^4$
$\hat\rho^8$	5	7	9	1	11	3	0	2	4	8	6	10	$\hat\rho^7$
$\hat\rho^3$	3	8	7	2	9	1	10	0	5	6	4	11	$\hat\rho^2$
$\hat\rho^7$	10	3	5	9	4	11	8	7	0	1	2	6	$\hat\rho^6$
$\hat\rho^9$	9	2	1	8	3	7	4	6	11	0	10	5	$\hat\rho^8$
$\hat\rho^2$	11	1	3	7	5	9	6	8	10	2	0	4	$\hat\rho$
$\hat\rho$	4	9	11	3	10	5	2	1	6	7	8	0	1
	$\hat\rho\phi$	$\hat\rho^9\phi$	$\hat\rho^2\phi$	$\hat\rho^3\phi$	$\hat\rho^7\phi$	$\hat\rho^8\phi$	$\hat\rho^5\phi$	$\hat\rho^{10}\phi$	$\hat\rho^6\phi$	$\hat\rho^4\phi$	$\hat\rho^{11}\phi$	ϕ	

TABLE 7.1. Webern's Matrix for H_1.

We can start with $\{0, 10, 3\}$ (in the set class $(0t5)$). Then:

$$\rho_2^5\phi_2(\{0,10,3\}) = \{5,8,9\},$$
$$\rho_2^3\phi_2(\{0,10,3\}) = \{1,4,2\},$$
$$\rho_2^{10}(\{0,11,2\}) = \{6,7,11\},$$

and hence we can choose the first row

$$r_0 = (0, 10, 3, 5, 9, 8, 4, 2, 1, 11, 6, 7).$$

This gives the Webern matrix shown in Table 7.2 on the facing page.

In this case, the hexachords $[0, 10, 8, 5, 3, 9]$ and $[4, 1, 11, 2, 6, 7]$ have, as in the original Webern matrix, three symmetries by transpositions (ρ_2^2, ρ_2^6, and ρ_2^{10}) and three symmetries by inversion ($\rho_2^3\phi_2$, $\rho_2^7\phi_2$, and $\rho_2^{11}\phi_2$).

	$\hat{\phi}$	$\hat{\rho}\hat{\phi}$	$\hat{\rho}^8\hat{\phi}$	$\hat{\rho}^5\hat{\phi}$	$\hat{\rho}^9\hat{\phi}$	$\hat{\rho}^4\hat{\phi}$	$\hat{\rho}^2\hat{\phi}$	$\hat{\rho}^7\hat{\phi}$	$\hat{\rho}^3\hat{\phi}$	$\hat{\rho}^6\hat{\phi}$	$\hat{\rho}^{10}\hat{\phi}$	$\hat{\rho}^{11}\hat{\phi}$	
1	0	10	3	5	9	8	4	2	1	11	6	7	$\hat{\rho}^{11}$
$\hat{\rho}^{11}$	7	0	2	8	3	1	10	11	4	5	9	6	$\hat{\rho}^{10}$
$\hat{\rho}^4$	8	5	0	9	10	3	11	7	2	6	4	1	$\hat{\rho}^3$
$\hat{\rho}^7$	2	3	1	0	8	7	9	4	6	10	5	11	$\hat{\rho}^6$
$\hat{\rho}^3$	1	8	7	3	0	2	5	6	11	9	10	4	$\hat{\rho}^2$
$\hat{\rho}^8$	3	9	8	10	5	0	6	1	7	4	11	2	$\hat{\rho}^7$
$\hat{\rho}^{10}$	6	7	11	1	2	4	0	5	10	8	3	9	$\hat{\rho}^9$
$\hat{\rho}^5$	5	11	10	6	4	9	2	0	3	7	1	8	$\hat{\rho}^4$
$\hat{\rho}^9$	9	6	5	4	11	10	7	8	0	1	2	3	$\hat{\rho}^8$
$\hat{\rho}^6$	11	2	4	7	1	6	3	10	9	0	8	5	$\hat{\rho}^5$
$\hat{\rho}^2$	4	1	6	2	7	11	8	9	5	3	0	10	$\hat{\rho}$
$\hat{\rho}$	10	4	9	11	6	5	1	3	8	2	7	0	1
	$\hat{\rho}\hat{\phi}$	$\hat{\rho}^2\hat{\phi}$	$\hat{\rho}^9\hat{\phi}$	$\hat{\rho}^6\hat{\phi}$	$\hat{\rho}^{10}\hat{\phi}$	$\hat{\rho}^5\hat{\phi}$	$\hat{\rho}^3\hat{\phi}$	$\hat{\rho}^8\hat{\phi}$	$\hat{\rho}^4\hat{\phi}$	$\hat{\rho}^7\hat{\phi}$	$\hat{\rho}^{11}\hat{\phi}$	$\hat{\phi}$	

TABLE 7.2. Webern's Matrix for H_2.

8. New Systems and Permutations

By Theorem 5.5, we know that all subgroups $H = \left\langle \hat{\rho}, \hat{\phi} \right\rangle$ of S_{12} that are isomorphic to D_{12} and contain a 12-cycle, i.e., the ones we use to construct new systems, are given by conjugates of D_{12} itself. This means that there is $\sigma \in S_{12}$ such that $\hat{\rho} = \sigma\rho\sigma^{-1}$ and $\hat{\phi} = \hat{\rho}^i \cdot (\sigma\phi\sigma^{-1})$ for some i. (Note that we cannot, in general, simply take $\hat{\phi} = \sigma\phi\sigma^{-1}$ due to our convention that $\hat{\phi}(0) = 0$. But we do have that $H = \langle \hat{\rho}, \sigma\phi\sigma^{-1} \rangle$.)

For instance, in the example of H_1 above, with $\rho_1 = (0\ \ 4\ \ 11\ \ 3\ \ 7\ \ 2\ \ 6\ \ 10\ \ 5\ \ 9\ \ 1\ \ 8)$ and $\phi_1 = \phi$, we can take

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 10 & 5 & 9 & 1 & 8 & 0 & 4 & 11 & 3 & 7 & 2 \end{pmatrix} \\ &= (0\ \ 6)(1\ \ 10\ \ 7\ \ 4)(2\ \ 5\ \ 8\ \ 11)(3\ \ 9). \end{aligned} \tag{8.1}$$

In general, since

$$\hat{\rho} = \sigma\rho\sigma^{-1} = (\sigma(0)\ \ \sigma(1)\ \ \sigma(2)\ \ \dots\ \ \sigma(11)),$$

the notion of interval using $\hat{\rho}$ can be obtained by permuting the original pitch classes. More precisely, since the pitch class interval from i to $i+1$ is 1, then in the new system given by $\hat{\rho} = \sigma\rho\sigma^{-1}$ (and its corresponding $\hat{\phi}$), the pitch class interval between $\sigma(i)$ and $\sigma(i+1)$ is 1. Thus, this new system can be seen as a *permutation* of the standard, where the role of the pitch class i is replaced by $\sigma(i)$.

The only pitch class that is somewhat special is 0, as we still use it as our first pitch class, and therefore it will affect the prime form of a set class: in the new system, the prime form is the one *starting at* 0 and most "concentrated to left" (in this new way of measuring intervals).

Note also that this special status of 0 only affects how we *represent* our set classes, but have no real effect on the structure.

Since these new systems are simply obtained by permutations, their intrinsic properties are the same as the standard system (with ρ and ϕ). For instance, comparing the Forte tables for D_{12} (Table 6.1 on page 33) and H_1 (Table 6.2 on page 33), one can see that they have the same number of rows and each row in one corresponds to an equivalent row in the other. For instance, the last row of the first (the row of (048)) corresponds to the eighth row on the second (the row of (075)).

Indeed, in the last row of the original we have the set class for the trichord $[0, 4, 8]$. Observe then that, with the σ_1 above that gives ρ_1, we have $\sigma_1(0) = 6$, $\sigma_1(4) = 1$, and $\sigma_1(8) = 11$ and the set class of $\{1, 6, 11\}$ has prime form (in the new system) (075), since $\rho_1^2(\{1, 6, 11\}) = \{7, 0, 5\}$. In a similar manner, one can check that the first row of the first table corresponds to the seventh row of the second one. Note that the class interval vector and number of symmetries match for the corresponding rows.

We shall refer to set classes from two systems that can be obtained by the corresponding permutation of pitch classes as *equivalent*. So, the set class (048) in the standard system is equivalent to the set class (075) in the system given by H_1.

	ϕ	$\hat{\rho}^{11}\phi$	$\hat{\rho}^{3}\phi$	$\hat{\rho}^{4}\phi$	$\hat{\rho}^{8}\phi$	$\hat{\rho}^{7}\phi$	$\hat{\rho}^{9}\phi$	$\hat{\rho}^{5}\phi$	$\hat{\rho}^{6}\phi$	$\hat{\rho}\phi$	$\hat{\rho}^{2}\phi$	$\hat{\rho}^{10}\phi$	
1	6	2	9	1	11	4	3	8	0	10	5	7	$\hat{\rho}^{10}$
$\hat{\rho}$	10	6	1	8	3	11	7	0	4	5	9	2	$\hat{\rho}^{11}$
$\hat{\rho}^{9}$	3	11	6	10	8	1	0	5	9	7	2	4	$\hat{\rho}^{7}$
$\hat{\rho}^{8}$	11	4	2	6	1	9	8	10	5	3	7	0	$\hat{\rho}^{6}$
$\hat{\rho}^{4}$	1	9	4	11	6	2	10	3	7	8	0	5	$\hat{\rho}^{2}$
$\hat{\rho}^{5}$	8	1	11	3	10	6	5	7	2	0	4	9	$\hat{\rho}^{3}$
$\hat{\rho}^{3}$	9	5	0	4	2	7	6	11	3	1	8	10	$\hat{\rho}$
$\hat{\rho}^{7}$	4	0	7	2	9	5	1	6	10	11	3	8	$\hat{\rho}^{5}$
$\hat{\rho}^{6}$	0	8	3	7	5	10	9	2	6	4	11	1	$\hat{\rho}^{4}$
$\hat{\rho}^{11}$	2	7	5	9	4	0	11	1	8	6	10	3	$\hat{\rho}^{9}$
$\hat{\rho}^{10}$	7	3	10	5	0	8	4	9	1	2	6	11	$\hat{\rho}^{8}$
$\hat{\rho}^{2}$	5	10	8	0	7	3	2	4	11	9	1	6	1
	$\hat{\rho}^{2}\phi$	$\hat{\rho}\phi$	$\hat{\rho}^{5}\phi$	$\hat{\rho}^{6}\phi$	$\hat{\rho}^{10}\phi$	$\hat{\rho}^{9}\phi$	$\hat{\rho}^{11}\phi$	$\hat{\rho}^{7}\phi$	$\hat{\rho}^{8}\phi$	$\hat{\rho}^{3}\phi$	$\hat{\rho}^{4}\phi$	ϕ	

TABLE 8.1. Another Webern's Matrix for H_1.

Similarly, applying σ_1 above to every single entry of the original Webern matrix (Table 3.1) gives a Webern matrix for the system determined by H_1, shown in Table 8.1. Note how in this case the transformations, shown on left, top, right, and bottom are exact matches between the two cases.

The example of H_2 can be obtained by taking

$$\sigma_2 = (0\ \ 1\ \ 8\ \ 7\ \ 6\ \ 9)(2\ \ 5\ \ 3\ \ 11\ \ 4),$$

as

$$\rho_2 = \sigma_2 \rho \sigma_2^{-1} = (0\ \ 10\ \ 4\ \ 1\ \ 8\ \ 5\ \ 11\ \ 2\ \ 3\ \ 9\ \ 6\ \ 7),$$
$$\sigma_2 \phi \sigma_2^{-1} = (0\ \ 11)(2\ \ 7)(3\ \ 6)(4\ \ 8)(5\ \ 10).$$

In this case, the standard choice of $\hat{\phi}$ (fixing 0) is

$$\phi_2 = \rho_1^6 \cdot (\sigma \phi \sigma^{-1}) = (1\ \ 9)(2\ \ 5)(3\ \ 8)(4\ \ 6)(7\ \ 10).$$

For those interested in using these new systems, the authors have routines for the math software Sage, along on instructions on how to use them, available at `https://github.com/lrfinotti/MusicalSystems`.

9. Applications I

In the next two sections we use the two systems described in Section 6, i.e., those given by H_1 and H_2, in concrete musical examples, namely those described in Section 4.

In this current section we will use these systems to rewrite the first eight bars of the first movement of Webern's *Concerto for Nine Instruments, Op. 24*, while in the next section we rewrite the first section of the song *Take a Bow* by Matthew Bellamy. These examples show how, despite the changes in the pitch classes, the interval relations and symmetries are preserved in the systems.

The first example, shown in Figure 9.1 on the facing page, rewrites the first eight bars of Webern's *Concerto for Nine Instruments, Op. 24* by simply permuting the pitch classes using σ_1 as given in Eq. (8.1), which is the permutation that gives the system determined by H_1. It corresponding matrix was given in Table 8.1.

Since all the pitch classes are simply permuted by σ_1 above, *all* its interval and symmetry properties are preserved in the new system (given by $H_1 = \sigma_1 D_{12} \sigma_1^{-1}$), and even the maps between the symmetric hexachords and trichords correspond to the original maps, as seen in Figure 9.2 on page 42. In other words, its analysis is completely equivalent to the original.

On the other hand, using the matrix from Table 7.1, which also uses the system given by H_1, we have a very different result. Following the same order of the series of the original composition, but with this different matrix, we produce the example given on 9.3 on page 43.

Figure 9.4 on page 44 shows the symmetries in this example. Note how the two hexachords that divide each series are from the set class $(04e6t5)$ and, in contrast to the example given by Figure 9.1, this set class is not equivalent to the hexatonic

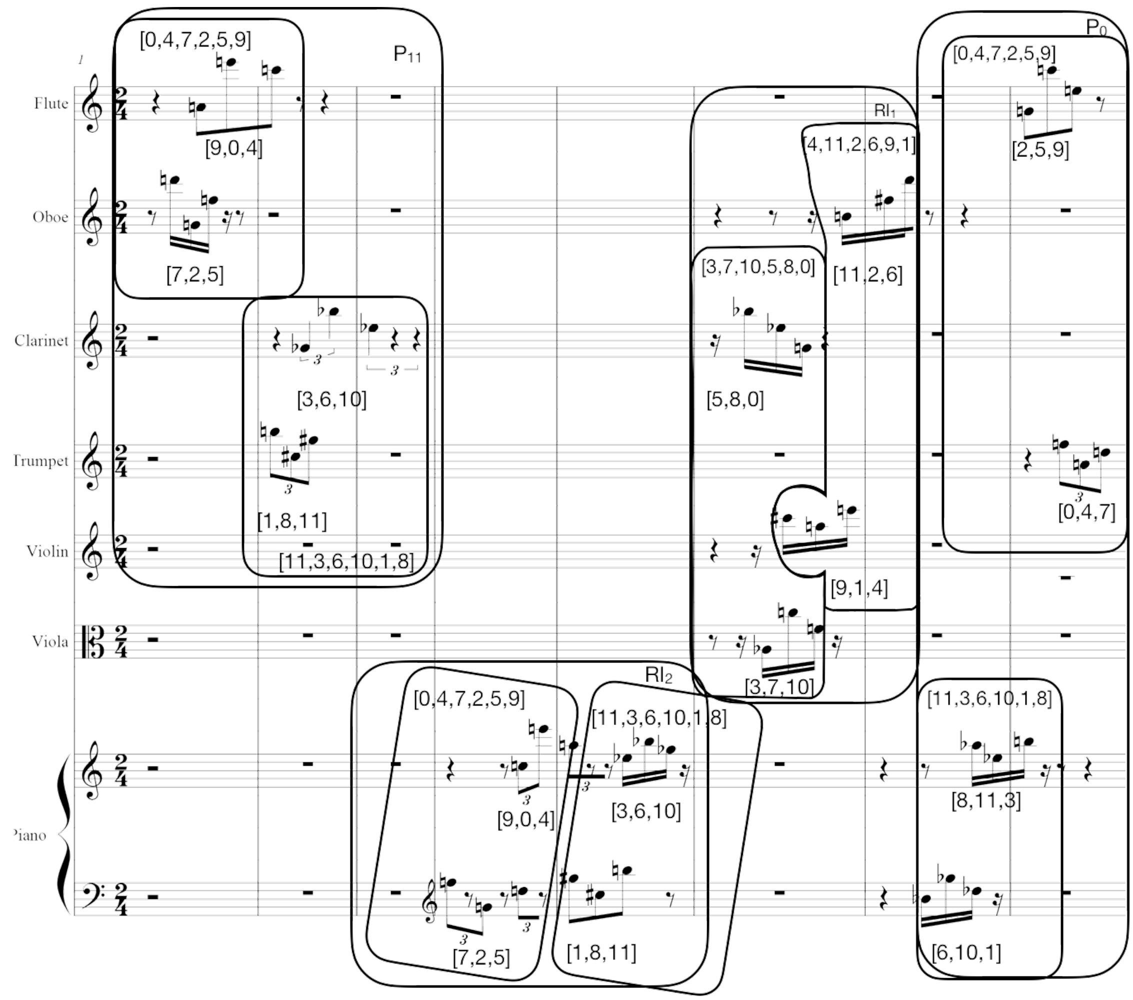

FIGURE 9.1. First eight bars of Webern's *Concerto for Nine Instruments* rewritten using σ_1.

collection, but to the set class (012678) from the standard system[6]. Although this set class is also symmetric, its members are mapped by only two transpositions and two inversions, as show in Figure 9.4.

The trichords that make these hexachrds are all from the set class (042), which are equivalent to the sec class (015) in the standard system. Thus, it is clear that indeed this example is not equivalent to the original.

We now give an example with the system given by the group H_2. This system is even more dissimilar to the standard system than H_1, as it not only changes operations of

[6] This hexachord is know as Messiaen's mode 5. See [5]

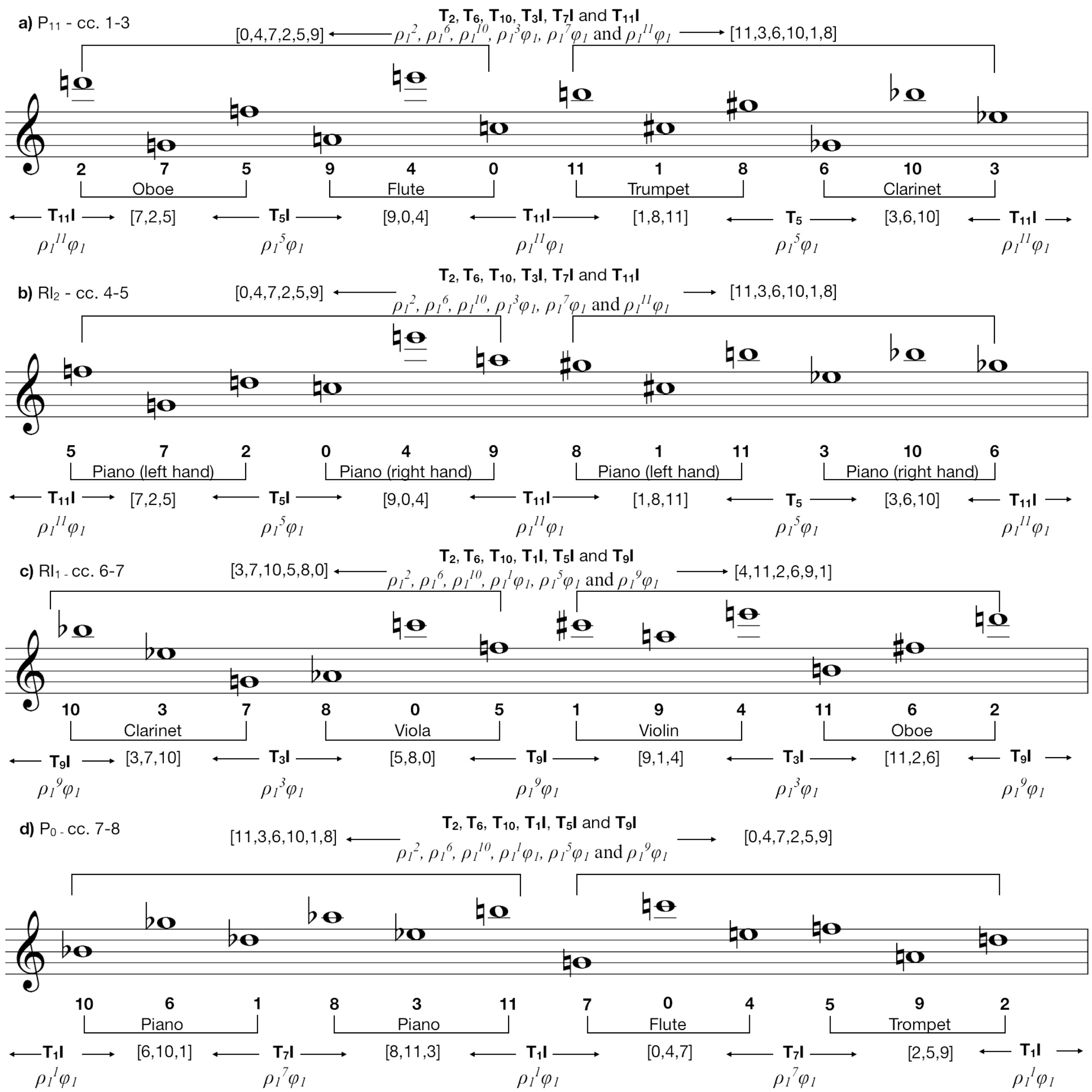

FIGURE 9.2. Symmetric operations relating the hexachords and trichords of the four series in the beginning of Webern's *Concerto*.

transposition (given by ρ), but also the operation of inversion (given ϕ). Figure 9.5 on page 45 shows the rewriting of Webern's piece using Table 7.2.

Although the matrix given in Table 7.2 is not equivalent to Webern's original matrix (given in Table 3.1), it does share some of the properties of the original, e.g., the two hexachords that divide the series, which are members of the set class $(0t8539)$, are related by three transpositions and three inversions (as observed in Section 7).

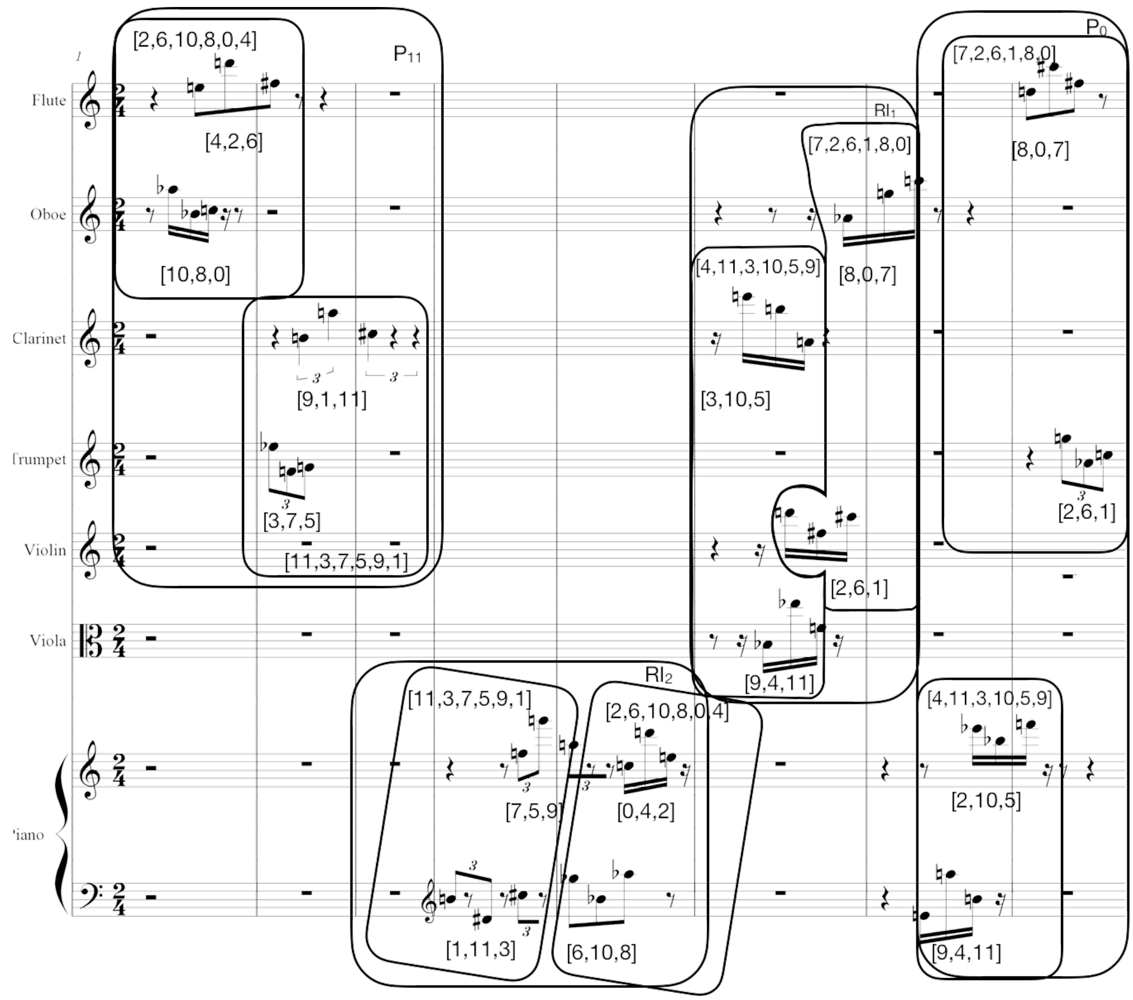

FIGURE 9.3. First eight bars of Webern's *Concerto for Nine Instruments* rewritten using Table 7.1.

The set class ($0t8539$) in this example is equivalent to the set class (014589) in the standard system, which is the same as in Webern's original. On the other hand, the trichords that divide ($0t8539$) in this new example are in the set class ($0t5$), which is *not* equivalent to the original set class (014) (as seen in Figure 4.2). It is in fact equivalent to (015) in the standard system.

Figure 9.6 on page 46 shows how the two hexachords that divide the series in this example are related by three inversions and three transpositions, as in the original. It also show how this new example differs from the original in that the original the

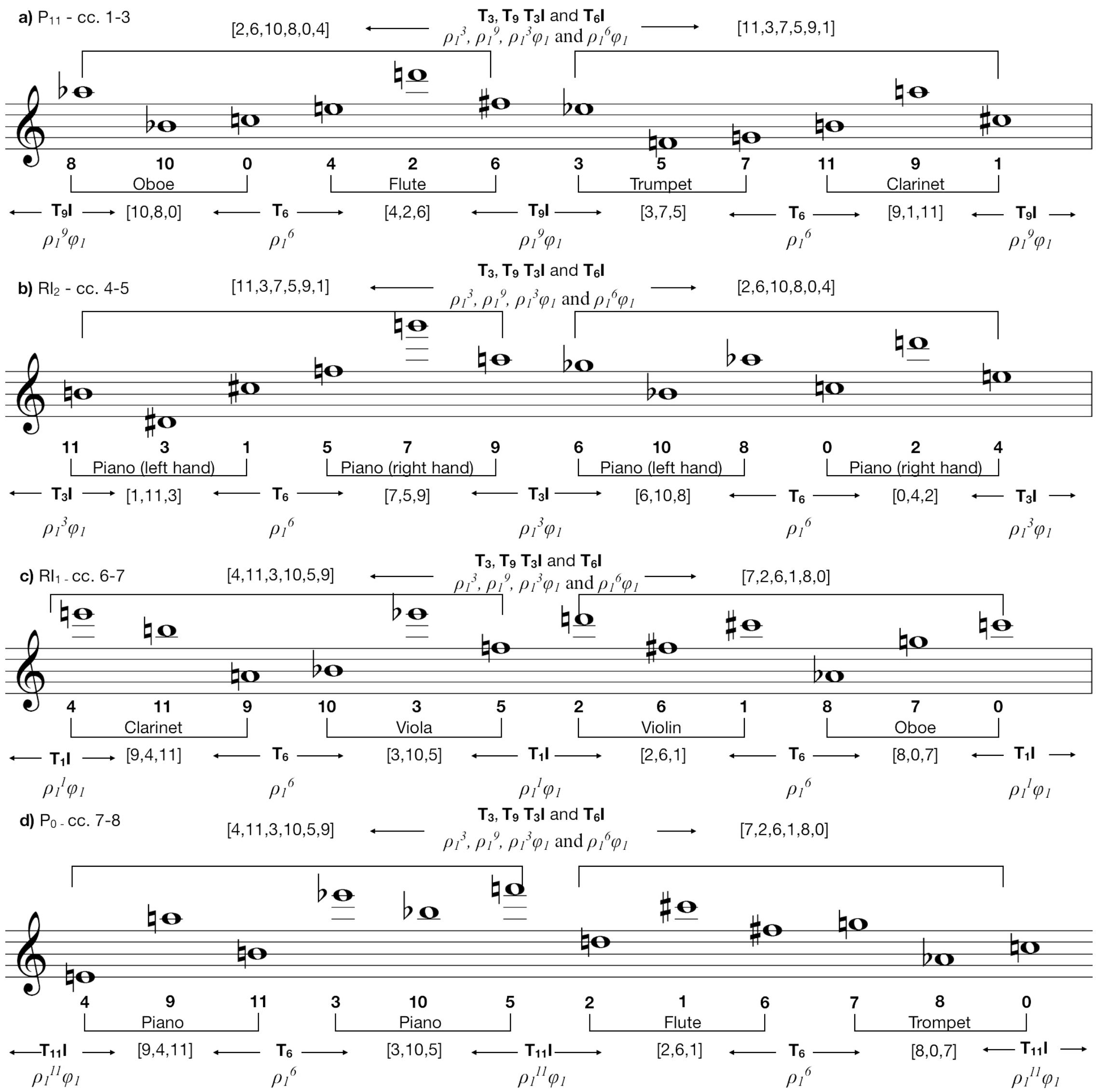

FIGURE 9.4. Symmetric operations relating the hexachords and trichords of the four series in the beginning of the example give in Figure 9.3.

relations between the trichords were always given by inversion operations, while in this new example the trichords are related alternatively by transpositions and inversions.

10. Applications II

We now apply our new systems to rewrite the first section of the song *Take a Bow* by Matthew Bellamy.

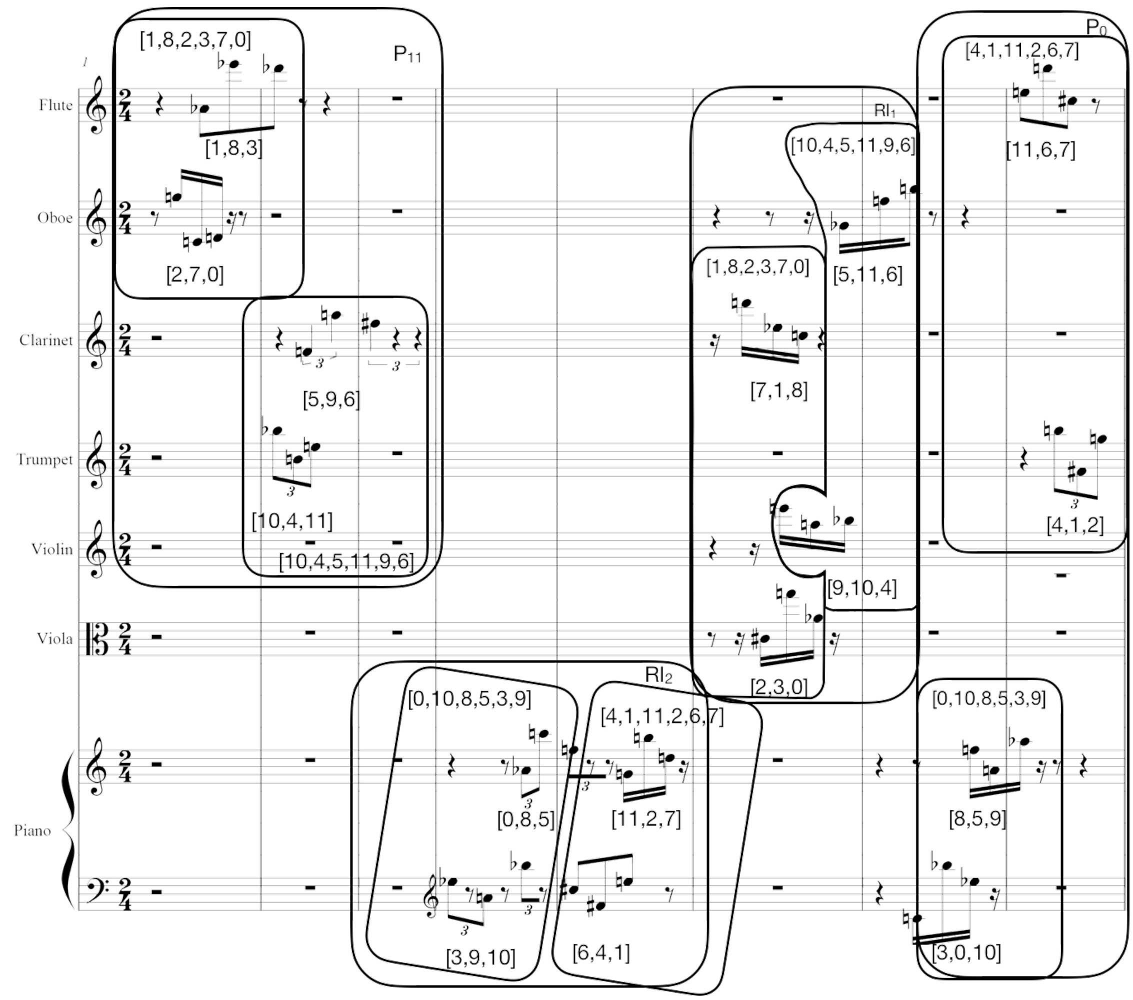

FIGURE 9.5. First eight bars of Webern's *Concerto for Nine Instruments* rewritten using Table 7.2.

As observed in Section 4, the harmony of this piece is composed of only major and minor triads, belonging to the set class (037), and augmented triads, belonging to the set class (048). Figure 4.3 also shows that the sequence of the members of the set class (037) is part of the chain $\langle \mathbf{PL'} \rangle$, where $\mathbf{P}$ is the Neo-Riemannian transformation that represents a contextual inversion in which the largest interval class between the triads is preserved, while the remaining interval class is changed by the smallest possible displacement (which is simply a semitone in the standard system), and $\mathbf{L'}$ is the Neo-Riemannian transformation that represents the contextual inversion in which the smallest interval class is displaced by the least possible interval, while the remaining interval class is preserved.

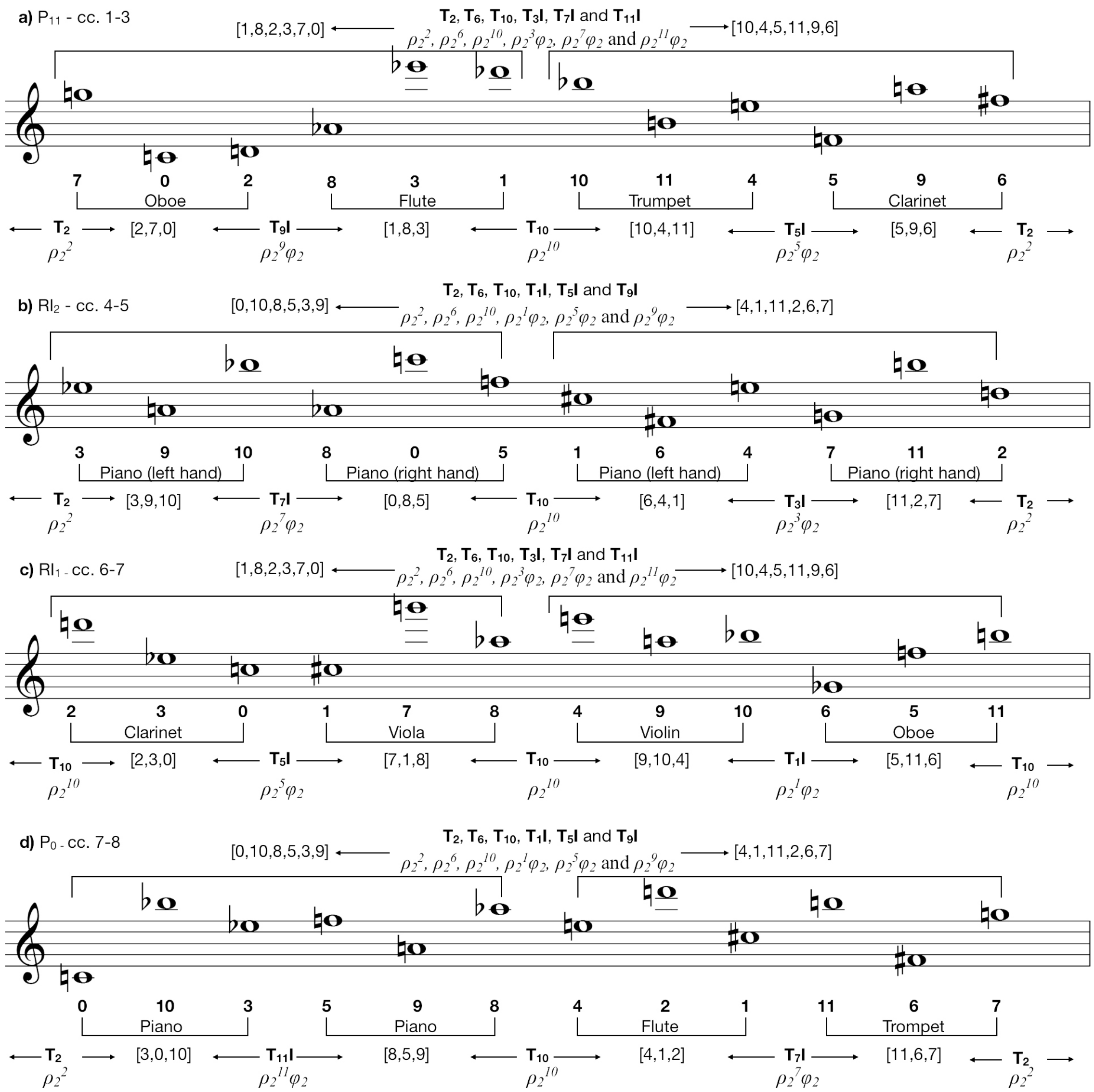

FIGURE 9.6. Symmetric operations relating the hexachords and trichords of the four series in the beginning of the example give in Figure 9.5.

In the standard system, the set class (037), which contains all major and minor triads, has interval vector 0001110, which is equivalent to the set class $(03t)$ in the system given by H_1. Figure 10.1 on the facing page shows the members of the chain $\langle \mathbf{PL'} \rangle$ for the set class $(03t)$ (in the new system) which will be used in the rewriting of the first section of the song.

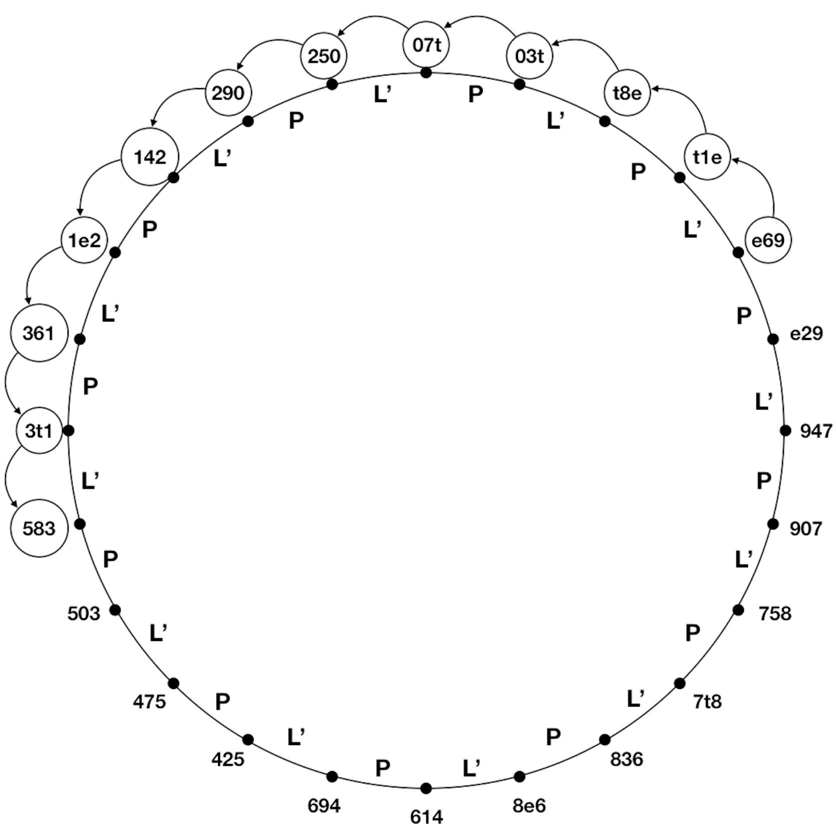

FIGURE 10.1. The cycle formed by the chain $\langle \mathbf{PL}' \rangle$ with members of $(03t)$ in the system given by H_1, emphasizing the sets used in the rewriting.

Observe how Figure 10.1 shows that the sets related by $\mathbf{P}$ and $\mathbf{L}'$ have the same coherence as in the standard system, i.e., in $\mathbf{P}$ the largest interval class 5 is preserved, while the remaining interval class is changed by a minimal interval, and in $\mathbf{L}'$ the smallest interval class 3 is changed by the smallest interval, while the remaining interval class is preserved. But, of course, in this new system this smallest interval is not (necessarily) a semitone, as intervals are measures according to the new ρ (in this case, $\rho_1 = (0\ \ 4\ \ 11\ \ 3\ \ 7\ \ 2\ \ 6\ \ 10\ \ 5\ \ 9\ \ 1\ \ 8)$).

For instance, note that the transformation between $[10, 1, 11]$ and $[10, 8, 11]$ is $\mathbf{P}$ as the pitch classes 10 and 11 form the largest interval class of the sets is maintained, while the pitch class 1 changed by the smallest interval to the pitch class 8.

Similarly, it can be observed that the transformation between $[11, 6, 9]$ and $[10, 1, 11]$ is $\mathbf{L}'$, as the pitch classes 6 and 9, which form the smallest interval are changed by single interval to 10 and 1, respectively, while the remaining pitch class remains unaltered.

Thus, one can see how the entire graph in Figure 10.1 is equivalent to the one in Figure 4.3 (which shows the chain $\langle \mathbf{PL}' \rangle$ in the standard system). The permutation of pitch classes in this case can be seen as

$$\begin{aligned}\sigma &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 4 & 11 & 3 & 7 & 2 & 6 & 10 & 5 & 9 & 1 & 8 \end{pmatrix} \\ &= (1\ \ 4\ \ 7\ \ 10)(2\ \ 11\ \ 8\ \ 5).\end{aligned}$$

Figure 4.4 showed the voice leading between the chords of the first section of the original song, and how an augmented triad was introduced between the consonant triads. As a consequence, the voice leading in this section is always made by preserving two pitch classes and changing the remaining pitch class by a smallest interval in every chord change.

In the system given by H_1 the set class equivalent to (048) (which contains the augmented triads) is (075). Figure 10.2 on the next page shows how the members of this set class, introduced between the members of $(03t)$ that are related by $\mathbf{L}'$, will play the same role of softening the voice leading, keeping only the displacement of one interval between every set.

In the same way we could trace of the path in which the voice leading occurs in the Cube Dance (Figure 4.5), we can do the same for the voice leading shown in Figure 10.2 if we adapt the Cube Dance to the system given by H_1. This is done by replacing the set classes by their equivalent set classes, i.e., replacing (048) by (075) and (037) by $(03t)$. This way, Figure 10.3 on page 50 gives a graph in this new system equivalent to the original Cube Dance, in which the voice leading between the sets connected by the edges always preserve two pitch classes, while the remaining is changed by a single interval.

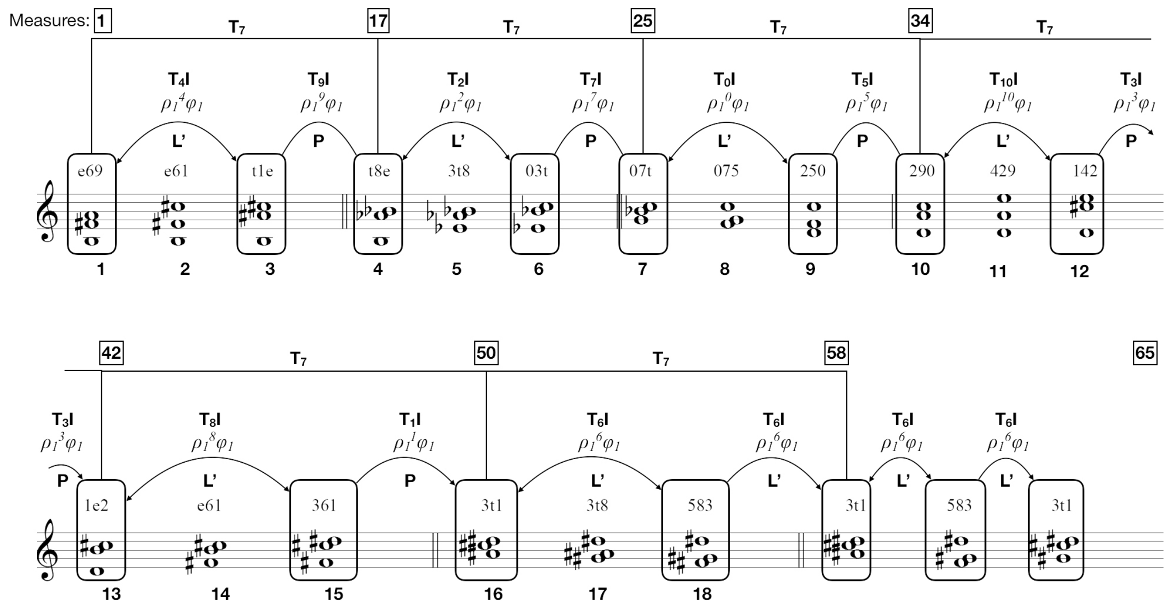

FIGURE 10.2. Voice leading of the first section of *Take a Bow* adapted to the system given by H_1.

Note that all triads from the original Cube Dance were replaced by the equivalent ones in the new system, and thus the path followed by the chord progression is the same in both examples. Moreover, note that with our definition for sum classes in these new systems (given in Section 6), we have that the sum classes of equivalent sets are the same, and hence the sets in the same voice-leading zone also have the same sum class (in the new sense).

Finally, we use the system given by H_2 to obtain a new rewrite the song. In this case the set class (037) in the standard system is equivalent to (012) in the new system. Figure 10.4 on page 51 shows the chain $\langle \mathbf{PL}' \rangle$ made by the elements of the set class (012), again with emphasis on those used in the rewriting.

In this new system, the set class (083) is the one equivalent to the set class of augmented triads (048) from the standard system. Again, the same movement of the voice-leading as in the original occurs in this example, as Figure 10.5 on page 52 shows.

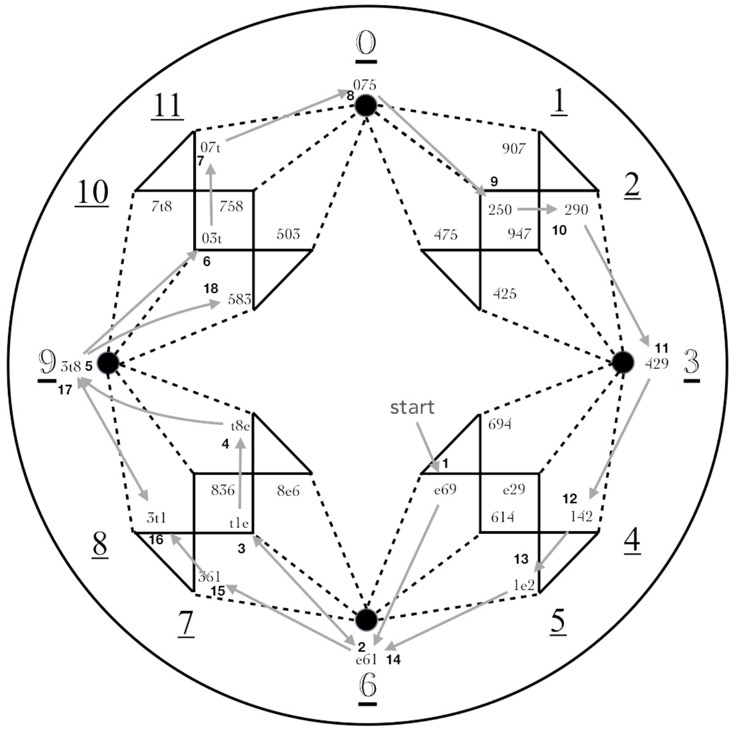

FIGURE 10.3. Sequence of all triads in section A of *Take a Bow* traced over the Cube Dance using H_1.

Finally, Figure 10.6 on page 52 shows the corresponding graph of the Cube Dance obtained with this rewriting, placing the members of (083) and (012) in place of the members of (048) and (037), their respective equivalent sets classes from the original.

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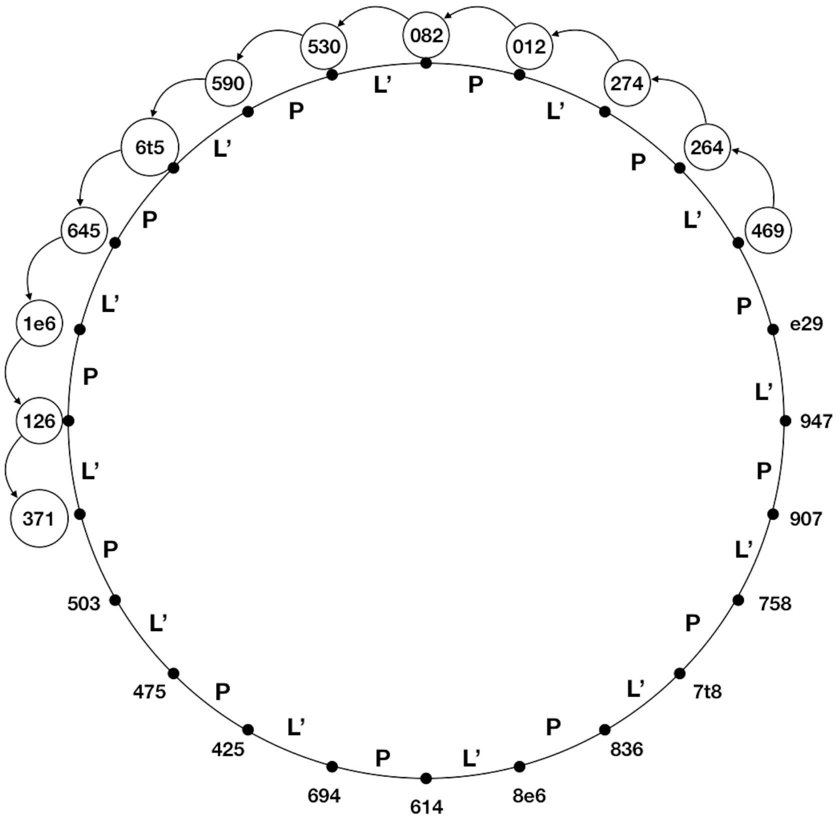

FIGURE 10.4. The cycle formed by the chain $\langle \mathbf{PL'} \rangle$ with members of (012) in the system given by H_2, emphasizing the sets used in the rewriting.

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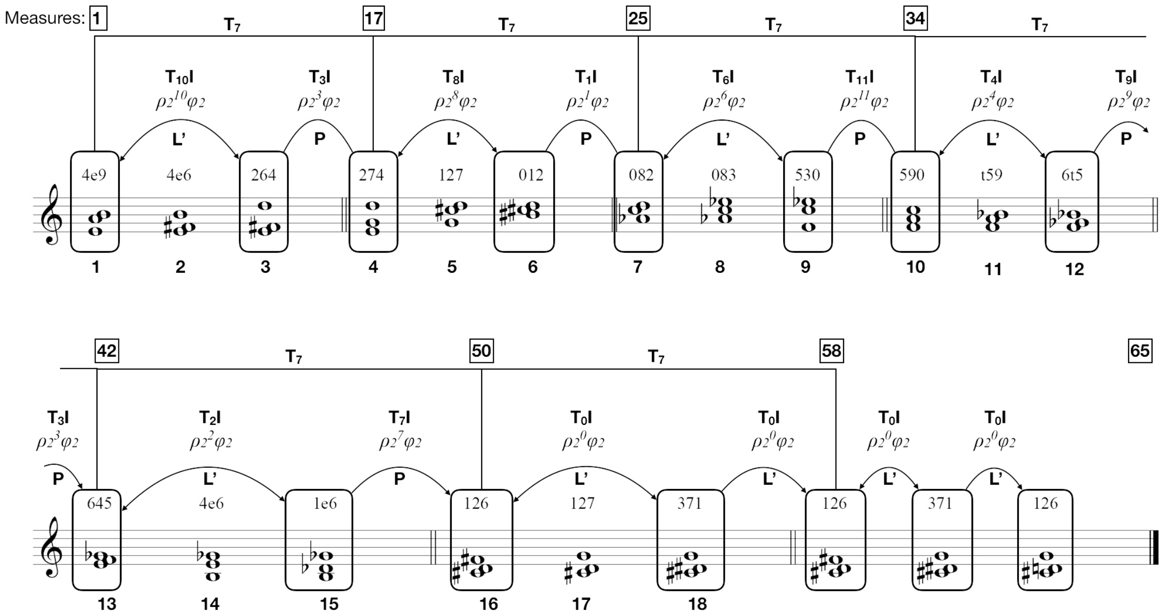

FIGURE 10.5. Voice leading of the first section of *Take a Bow* adapted to the system given by H_2.

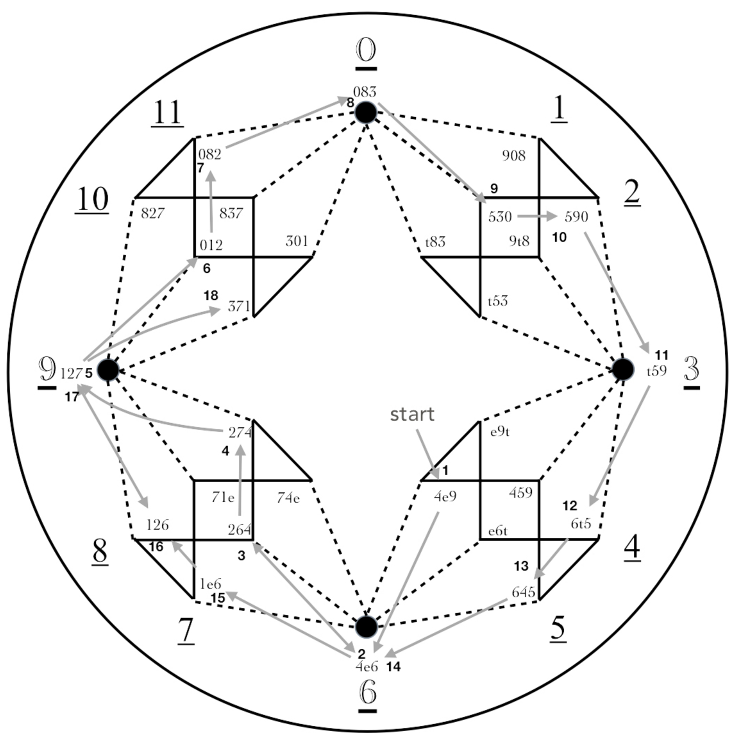

FIGURE 10.6. Sequence of all triads in section A of *Take a Bow* traced over the Cube Dance using H_2.

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