

# Math 456

Luís Finotti  
Spring 2007

Name: .....

Student ID (last 5 digits): XXXX.....

## MIDTERM 2

You have 50 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last five digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 5 questions and 7 printed pages (including this one and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

**Show all work!** Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions.**

**Good luck!**

Question	Max. Points	Score
1	12	
2	22	
3	22	
4	22	
5	22	
Total	100	

1) Prove that if  $f \in \mathbb{Z}[x]$  is primitive and  $g \in \mathbb{Z}[x]$  divides  $f$  in  $\mathbb{Z}[x]$ , then either  $g$  or  $-g$  is also primitive.

2) Find whether or not the following polynomials are irreducible over  $\mathbb{Q}[x]$ .

(a)  $f_1(x) = x^4 + x^3 + x - 6$

(b)  $f_2(x) = x^6 - 2x^5 + 14x^2 - 8x + 34$

(c)  $f_3(x) = 100x^3 - x + 2008$

(d)  $f_4(x) = x^4 + x^3 + x^2 + x + 1$

**3)** Let  $F$  be a field. We say that  $\alpha \in F$  is a *multiple root* of  $f(x) \in F[x]$  if  $f(x) = (x - \alpha)^2 \cdot g(x)$ , for some  $g \in F[x]$ .

(a) Prove that if  $\alpha$  is a multiple root of  $f$ , then  $f(\alpha) = f'(\alpha) = 0$ , where  $f'(x)$  is the derivative of  $f(x)$  [as in calculus]. [Note that all calculus formulas for derivatives hold for polynomials.]

(b) Prove that if  $f(x) \in F[x]$  is irreducible, then  $f(x)$  has no multiple roots in any extension of  $F$ , as long as  $f'(x) \neq 0$ . [**Hint:** What's the greatest common divisor of  $f(x)$  and  $f'(x)$ ?]

4) Let  $R$  be a UFD and let  $P$  be a non-zero *prime* ideal of  $R$  such that if  $P'$  is another prime ideal, with  $(0) \subsetneq P' \subseteq P$ , then  $P' = P$ . Prove that  $P$  is principal. [**Hint:**  $P = (p)$  is prime iff  $p$  is what?]

5) Maximal ideals of polynomial rings with complex coefficients.

- (a) Prove that if  $I$  is an ideal of  $\mathbb{C}[x, y]$  and  $M$  is a maximal ideal containing  $I$ , then there is a point  $(a, b)$  such that for all  $f(x, y) \in I$ , we have  $f(a, b) = 0$ .

[**Observation:** This statement is also true for  $n$  variables (with an analogous solution).]

- (b) Let  $I = (3x - y - 2, y - x^2)$  be an ideal of  $\mathbb{C}[x, y]$ . Find *all* maximal ideals of  $\mathbb{C}[x, y]$  that contain  $I$ .

**Scratch:**