Extra Credit 2

Math 456

April 4, 2007

1. Let F be a field and

$$F[[x]] \stackrel{\text{def}}{=} \left\{ \sum_{n=0}^{\infty} a_n x^n \ : \ a_n \in F \right\},$$

i.e., the *ring of power series over* F. This is indeed an *integral domain*, with the sum and product defined as expected:

$$\left[\sum_{n=0}^{\infty} a_n x^n \right] + \left[\sum_{n=0}^{\infty} b_n x^n \right] \stackrel{\text{def}}{=} \left[\sum_{n=0}^{\infty} (a_n + b_n) x^n \right]$$

and

$$\left[\sum_{n=0}^{\infty} a_n x^n \right] \cdot \left[\sum_{n=0}^{\infty} b_n x^n \right] \stackrel{\text{def}}{=} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \right]$$

[*You don't have to prove any of the above!!*] Let $\sigma : F[[x]] - \{0\} \to \{0, 1, 2, \dots\}$ be defined as: $\sigma\left(\sum_{n=0}^{\infty} a_n x^n\right)$ is the *smallest* n such that $a_n \neq 0$.

In this problem we will prove that $F[[x]]$ is a Euclidean domain.

(a) Prove that $F[[x]]^{\times} = \{a \in F[[x]] \ : \ \sigma(a) = 0\}$.

(b) Prove that for all $a \in F[[x]]$, we can write $a = x^{\sigma(a)} a'$, where $a' \in F[[x]]^{\times}$.

(c) Use the above to prove that $a \mid b$ in $F[[x]]$ iff $\sigma(a) \leq \sigma(b)$.

(d) Prove that $F[[x]]$ is a Euclidean domain [with size function σ].