Final (Take Home)

M552 – Abstract Algebra

May 6th, 2008

1. [20 points] Let K/F be an algebraic extension such that every non-constant $f(x) \in F[x]$ has at least one root in K. Prove that K is algebraically closed [and hence K is the algebraic closure of F].

[**Hint:** Be careful with inseparability. Also, the *Primitive Element Theorem* might be useful.]

Proof. It suffices to show that $\bar{F} \subseteq K$ [as K/F is algebraic].

Let $\alpha \in \bar{F}$ and $f \stackrel{\text{def}}{=} \min_{\alpha,F}$. Let E be the splitting field of f over F [and hence E/F is normal]. By Proposition V.6.11 of Lang's book, we have that $E = E_{\text{sep}}E_{\text{pi}}$, with E_{sep}/F normal and separable and E_{pi}/F purely inseparable. Moreover, the

Now, E_{sep}/F is finite [since $[E : F] \leq (\deg f)!$] and separable, so $E_{\text{sep}} = F[\beta]$. Let $g \stackrel{\text{def}}{=} \min_{\beta,F}$. Then, g has a root $\gamma \in K$ by assumption. But E_{sep}/F is normal, hence $\gamma \in E_{\text{sep}} = F[\beta]$. But, since $[F[\gamma] : F] = \deg g = [F[\beta] : F]$, and hence, $E_{\text{sep}} = F[\beta] = F[\gamma] \subseteq K$.

Also, since E_{ip}/F is purely inseparable, for all $\beta \in E_{\text{ip}}$, we have that β is the only solution of its minimal polynomial over F, since it's of the form $x^{p^n} - \beta^{p^n}$ [as seen in pg. 249 of Lang]. Hence, $\beta \in K$ and thus, $E_{\text{ip}} \subseteq K$.

So, we have that $\alpha \in E = E_{\text{sep}}E_{\text{pi}} \subseteq K$. Since α was arbitrary, $\bar{F} \subseteq K$.

□

2. [20 points] Let K/F be an algebraic, but *infinite* extension. Let $G \stackrel{\text{def}}{=} \text{Aut}(K/F)$. Prove that G is *residually finite*, i.e., that the intersection of all subgroups of G which are normal and of finite index [in G] is $\{1\}$.

Proof. Let $\sigma \in G - \{1\}$. Then, there exists $\alpha_1 \in K$ such that $\alpha_2 \stackrel{\text{def}}{=} \sigma(\alpha_1) \neq \alpha_1$. Let $f \stackrel{\text{def}}{=} \min_{\alpha,F}$. Note that α_2 is also a root of f, since it's irreducible and σ fixes F.

Let $\Omega \stackrel{\text{def}}{=} \{\alpha_1, \ldots, \alpha_k\}$ be all the roots of f in K. [Since K/F might not be normal, maybe we don't have all of them.] Since all $\tau \in G$ fixes F [and takes K to K], we have $\tau|_\Omega$ induces a permutation of Ω. So, we have $\phi : G \to S_\Omega = S_k$, given by $\phi(\tau) = \tau|_\Omega$.

Let $H \stackrel{\text{def}}{=} \ker \phi$. Then, $H \lhd G$. Moreover, since $|G : H| = |\phi(G)| \leq |S_k| = k!$ [as $G/H \cong \phi(G)$], we have that H has finite index.

Now, observe that $\sigma \notin H$, since $\phi(\alpha_1) = \alpha_2$, and hence $\phi(\sigma) \neq 1$. So, for all $\sigma \in G - \{1\}$, there is $H \lhd G$ of finite index such that $\sigma \notin H$, and therefore, the intersection of all normal subgroups of finite index is $\{1\}$.

□