1) [20 points] Mark each statement as true (T) or false (F). [No need to justify your answers here.]

(a) For all $n \times n$ matrices A and B, we have $\det(A + B) = \det(A) + \det(B)$.

Answer: False. [See pg. 103.]

(b) For all $n \times n$ matrices A and B, we have $\det(A \cdot B) = \det(A) \cdot \det(B)$.

Answer: True.

(c) If E is an elementary matrix, then $\det(E) = 1$.

Answer: False. You can multiply a row by $k \neq 1$, obtaining detarminant k.

(d) If A is a *singular* $n \times n$ matrix, then the system $A \cdot \mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Answer: False. It can also have no solution [depending on $\mathbf{b}$].

(e) For all $m \times n$ matrices A and B and scalar k, we have $kB + kA = k(A + B)$.

Answer: True.

(f) Every homogeneous system has a solution.

Answer: True. [For example, $x_1 = \cdots = x_n = 0$ is always a solution.]

(g) Every system with more variables than equations has infinitely many solutions.

Answer: False. For example:

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

has no solution.

(h) For all $n \times n$ matrices A and scalar k, we have that $\det(kA) = k \det(A)$.

Answer: False. The correct formula is $\det(kA) = k^n \det(A)$.

(i) For all $n \times n$ matrices A and B, we have $(A \cdot B)^{\mathrm{T}} = A^{\mathrm{T}} \cdot B^{\mathrm{T}}$.

Answer: False. The correct formula is $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

(j) If A is a square matrix and $A\mathbf{x} = \mathbf{b}$ has a single solution, then so does $A^{\mathrm{T}}\mathbf{x} = \mathbf{b}$.

Answer: True, as the system has a single solution if and only if $\det(A) \neq 0$, and then $\det(A^{\mathrm{T}}) = \det(A) \neq 0$.

2) [20 points] *Quickies!* [These should take you 10 seconds each if you've studied.] You don't need to justify your answers.

(a) $\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}^{-1} =$

Answer: $= \dfrac{1}{6-7}\begin{bmatrix} 3 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}.$

(b) $\begin{vmatrix} -3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ -2 & 2 & 4 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} =$

Answer: $= (-3)\cdot 1 \cdot 2 \cdot 3 \cdot 1 = -18$ [detarminant of lower triangular matrix].

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} =$

Answer: $= \begin{bmatrix} 2 & 2 & 2 \\ -2 & -2 & -2 \\ 15 & 15 & 15 \end{bmatrix}$ [multiplication by diagonal matrix].

(d) If $E \cdot \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 5 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 5 & -4 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, then $E =$

Answer: $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ [multiplication by elementary matrix].

(e) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, then $\begin{vmatrix} 2g & 2h & 2i \\ d+3a & e+3b & f+3c \\ -a & -b & -c \end{vmatrix} =$

Answer: $= (-1)\cdot(-1)\cdot 1 \cdot 2 \cdot 2 = 4$ [(switch 1st and 3rd rows) + (multiply row by 2) + multiply row by -1) + (add -3 times the 3rd row to the second)].

3) [15 points] Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ 0 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Compute $A^{\mathrm{T}} \cdot B \cdot \mathrm{tr}(C)$.

Solution. We have

$$A^{\mathrm{T}} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

and $\mathrm{tr}(C) = 2 + 1 = 3$. Hence,

$$\begin{aligned} A^{\mathrm{T}} \cdot B \cdot \mathrm{tr}(C) &= \begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ 0 & 5 \end{bmatrix} \cdot 3 \\ &= \begin{bmatrix} -2 & 17 \\ 5 & 7 \\ 8 & 1 \end{bmatrix} \cdot 3 \\ &= \begin{bmatrix} -6 & 51 \\ 15 & 21 \\ 24 & 3 \end{bmatrix} \end{aligned}$$

$\square$

4) [20 points] Let

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -3 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 2 & 6 \end{bmatrix}.$$

How many solutions does the system $A\mathbf{x} = \mathbf{0}$ have? [I'm not asking you to solve it, just for the number of solutions.]

Solution. We just need to find the detarminante. [If it is not zero, then the system has only one solution, and it will have infinitely many solutions otherwise.

We have:

$$\det(A) = 1 \cdot \begin{vmatrix} 4 & 1 & 0 & 3 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 6 \end{vmatrix} \qquad \text{[cofactors through the 2nd row]}$$

$$= (-1) \cdot \begin{vmatrix} 4 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 6 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 1 & 3 \\ 1 & -1 & -3 \\ 1 & 2 & 6 \end{vmatrix} \qquad \text{[cofactors through the 3rd col]}$$

$$= (-1) \cdot (24 + 12 - (3 + 12)) + 2 \cdot 0 \qquad \text{[Sarrus rule and col. mult. of another]}$$

$$= -21.$$

Hence the system has only one solution [the trivial $x_1 = \cdots = x_5 = 0$].

□

5) [25 points] Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Find $\det(A^3)$, A^{-1} [if possible; if not, justify], and solve $A\mathbf{x} = \mathbf{b}$. [Don't work too hard here! Think a little before you start!]

Solution. We find the inverse and solve the system at the same time by putting the coefficient matrix in reduced echelon form. We also keep track of the elementary row operations so that we can find $\det(A)$.

$$\left[\begin{array}{ccc|ccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & -1 & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc|c} 1 & 1 & 0 & 1/2 & 1 & -1/2 & 2 \\ 0 & 1 & 0 & -1/2 & 2 & -1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1/2 & 2 & -1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 & -1 \end{array}\right]$$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

and the system has a single solution

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

The operations that change the determinant are switching rows [which switches the sign of the determinant] and multiplying a row by a scalar [which divides the determinant by that scalar]. We switched two rows once, mutiplied a row by -1 once and by $1/2$ once. Thus, $\det(A) = 2$. Thus, $\det(A^3) = (\det(A))^3 = 2^3 = 8$.

□