1) [10 points] Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (3, k)$. Find k such that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/4$.

Solution. We have:

$$(1,1) \cdot (3,k) = \|(1,1)\| \, \|(3,k)\| \cos(\pi/4).$$

So,

$$3 + k = \sqrt{2}\,\sqrt{9 + k^2}\,\sqrt{2}/2 = \sqrt{k^2 + 9}$$

Squaring:

$$k^2 + 6k + 9 = k^2 + 9,$$

and hence $6k = 0$, i.e., $k = 0$.

Now, since we've squared the equation, we need to check that we've got it right [as squaring "$1 = -1$" gives a true statement], but indeed,

$$\frac{(1,1) \cdot (3,0)}{\sqrt{2} \cdot 3} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos(\pi/4).$$

[To see why this is necessary, replace $(3, k)$ by $(-3, k)$. The algebra will give you $k = 0$, but the fact is that there is no k that will make the angle $\pi/4$.]

[Note that this could be easily seen geometrically.]

□

2) [20 points] Give the matrix that represents the following linear transformation (from $\mathbb{R}^3$ to $\mathbb{R}^3$): projection to the xy-plane, followed by a rotation of $\pi/2$ around the z-axis, followed by a reflection on the yz-plane.

Solution. We find the matrix by checking what it does to the standard basis vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$.

$$\mathbf{e}_1 \longrightarrow \mathbf{e}_1 \longrightarrow \mathbf{e}_2 \longrightarrow \mathbf{e}_2,$$

also,

$$\mathbf{e}_2 \longrightarrow \mathbf{e}_2 \longrightarrow -\mathbf{e}_1 \longrightarrow \mathbf{e}_1,$$

and finally,

$$\mathbf{e}_3 \longrightarrow \mathbf{0} \longrightarrow \mathbf{0} \longrightarrow \mathbf{0}.$$

So, the matrix is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Alternatively, the matrices for the three transformations are, respective:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence the matrix of the linear transformation we are looking for is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

3) [15 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(2,1) = (3,0)$ and $T(0,1) = (1,-2)$. Find the inverse of T or show that such inverse does not exist.

Solution. We have that the first column of $[T]$ is $T(1,0)$, while the second column is $T(0,1)$. Hence, the statement gives us the second column. To find the first, we have [since T is linear], we have that

$$(3,0) = T(2,1) = T(2\,(1,0) + (0,1)) = 2\,T(1,0) + T(0,1) = 2\,T(1,0) + (1,-2).$$

Solving for $T(1,0)$, we have $T(1,0) = 1/2\,(2,2) = (1,1)$, and hence,

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Then T is invertible if, and only if, the matrix $[T]$ is invertible, which is true:

$$\left[T^{-1}\right] = [T]^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus,

$$T^{-1}(x,y) = \left(\frac{2x+y}{3}, \frac{x-y}{3}\right).$$

□

4) [15 points] Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and assume that 0 is an eigenvalue of T. Can T be one-to-one? Can it be onto? Justify your answers!

Proof. No [for both]. Saying that 0 is an eigenvalue is the same as to say that there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $T(\mathbf{v}) = 0 \cdot \mathbf{v} = \mathbf{0}$. Hence, since always for a linear transformation $T(\mathbf{0}) = \mathbf{0}$, we have two different vectors [namely, $\mathbf{v}$ and $\mathbf{0}$] with the same value by T [namely, $\mathbf{0}$], and hence T is not one-to-one.

Now, since the domain and codomain of T are equal [namely, $\mathbb{R}^n$], we have that T is one-to-one if, and only if, it is onto. Hence, since T is not one-to-one, it cannot be onto.

□

5) [15 points] Let $V = \mathbb{R}^2$ with the usual sum of vectors in $\mathbb{R}^2$, but with the following multiplication by real numbers: $k(x, y) = (ky, kx)$. Show that V is *not* a vector space.

Solution. We have

$$1\,(1, 0) = (0, 1),$$

and thus, property 8 [from the list given below] fails. Hence, it cannot be a vector space.

[You could also show instead the property 7 fails:

$$(2 \cdot 3)\,(1, 0) = 6\,(0, 1) = (0, 6),$$

while

$$2\,(3\,(1, 0)) = 2\,(0, 3) = (6, 0).$$

But, you only need to show that one fails.]

□

6) [10 points] Consider the set of all matrices

$$\begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix}, \qquad a, b \in \mathbb{R}.$$

with:

$$\begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix} + \begin{bmatrix} a' & 1 \\ 2 & b' \end{bmatrix} = \begin{bmatrix} a+a' & 1 \\ 2 & b+b' \end{bmatrix}, \qquad k \begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 2 & kb \end{bmatrix}.$$

This set with this sum and scalar multiplication *is* a vector space. [You do not need to prove it! Just take my word for it.] What is the zero of this vector space? What is

$$-\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}?$$

Solution. If

$$0 = \begin{bmatrix} x & 1 \\ 2 & y \end{bmatrix},$$

we must have that

$$\begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix} + \begin{bmatrix} x & 1 \\ 2 & y \end{bmatrix} = \begin{bmatrix} a+x & 1 \\ 2 & b+y \end{bmatrix} = \begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix}.$$

Thus, we must have $x = y = 0$, i.e.,

$$0 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

If

$$-\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} x & 1 \\ 2 & y \end{bmatrix},$$

we must have that

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 2 & y \end{bmatrix} = \begin{bmatrix} 3+x & 1 \\ 2 & -1+y \end{bmatrix} = 0 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Thus, we must have $x = -3$ and $y = 1$, i.e.,

$$-\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}.$$

□

7) [15 points] Let V be the set of all functions [from $\mathbb{R}^2$ to $\mathbb{R}^2$] of the form $f(x,y) = (ax+by, ax^2+by^2)$, where $a,b \in \mathbb{R}$. Is V a vector space with the usual sum and scalar multiplication of functions? [Justify!]

Solution. The set of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, since $\mathbb{R}^2$ is a vector space, is itself a vector space [which I denoted in class by $F(\mathbb{R}^2,\mathbb{R}^2)$]. Since V is inside this vector space [and we are using the same operations], it suffices to show that it is a subspace.

If $f(x,y) = (ax+by, ax^2+by^2)$ and $g(x,y) = (a'x+b'y, a'x^2+b'y^2)$, then

$$f(x,y)+g(x,y) = ((a+a')x+(b+b')y, (a+a')x^2+(b+b')y^2),$$

and hence $f(x,y)+g(x,y) \in V$.

Also, given any $k \in \mathbb{R}$,

$$kf(x,y) = ((ka)x+(kb)y, (ka)x^2+(kb)y^2),$$

and hence $kf(x,y) \in V$.

These two properties gives us that V is then a vector space [more precisely, a subspace of $F(\mathbb{R}^2,\mathbb{R}^2)$]. [Note that V is not empty, as $f(x,y) = (0,0) \in V$.]

□

Vector Space Requirements

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u}+\mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.