1) [10 points] Put the following matrix in *reduced* row echelon form:

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned}
\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ -1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

□

2) You should be able to answer the following questions *quickly*. You do *not* need to justify your answers.

(a) [4 points] Compute $\det\left(\begin{bmatrix} 1 & 0 & 2 & -3 \\ 1 & 2 & -1 & 0 \\ 1 & 4 & -2 & 2 \\ 3 & 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}\right)$.

Solution. Since $\det(A \cdot B) = \det(A) \cdot \det(B)$ and the determinant of the second matrix is zero [as it has two proportional rows], we get that the determinant is zero. $\square$

(b) [3 points] Give the matrix that represents the reflexion on the xy-plane in $\mathbb{R}^3$.

Solution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\square$

(c) [3 points] Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (2x_1 - 3x_2, 0, x_2 - x_3, x_1).$$

Give $[T]$ [i.e., the matrix associated to this linear transformation].

Proof.

$$[T] = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

□

(d) [4 points] If $W = \text{span}(\{(1, 2, -3, 1), (0, 2, 0, 2), (-1, 1, 3, 4)\})$, then the orthogonal complement of W given by what matrix space [i.e., row space, column space, or null space] of the what matrix?

Solution. Nullspace of $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 2 \\ -1 & 1 & 3 & 4 \end{bmatrix}$ [i.e., vectors as rows]. □

(e) [3 points] Let T_A be the a linear transformation associated to the m by n matrix A. If T_A is one-to-one, then what can we say about the rank of A? [If this rank is unrelated to whether or not T_A is one-to-one, just say so.]

Solution. We must have $\operatorname{rank}(A) = n$. □

(f) [4 points] Let A be a 3 by 3 matrix with eigenvalues -2 and 1, with their respective eigenspaces being $\operatorname{span}\{(1,0,-1),(1,1,1)\}$ and $\operatorname{span}\{(0,0,1)\}$. Give the matrix P such that $P^{-1}AP$ is diagonal, as well as $P^{-1}AP$ itself.

Solution.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

3) [10 points] Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a matrix B such that

$$\left(A^{\mathrm{T}} + 2B\right)^{-1} = C.$$

Solution. We have, $A^{\mathrm{T}} + 2B = C^{-1}$, and so $B = \frac{1}{2}(C^{-1} - A^{\mathrm{T}})$. So,

$$A = \frac{1}{2}\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}.$$

$\square$

4) [10 points] Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Show that the set W of all vectors $\mathbf{w}$ in $\mathbb{R}^n$ such that $\mathbf{v} \cdot \mathbf{w} = 0$ is a subspace of $\mathbb{R}^n$. [**Note:** Part of this is to show that W is non-empty. To show this you just need to find a vector that you can guarantee is in W.]

Solution. First observe that $\mathbf{0} \in W$, as $\mathbf{v} \cdot \mathbf{0} = 0$. [Hence, W is non-empty.]

Suppose that $\mathbf{w}_1, \mathbf{w}_2 \in W$ [i.e., $\mathbf{v} \cdot \mathbf{w}_1 = 0$ and $\mathbf{v} \cdot \mathbf{w}_2 = 0$]. Then,

$$\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \cdot \mathbf{w}_1 + \mathbf{v} \cdot \mathbf{w}_2 = 0 + 0 = 0$$

Also, for any $k \in \mathbb{R}$,

$$\mathbf{v} \cdot (k\mathbf{w}_1) = k(\mathbf{v} \cdot \mathbf{w}_1) = k \cdot 0 = 0.$$

Thus, sums and scalar products of elements in W are in W, and hence [since W is non-empty] W is a subspace of $\mathbb{R}^n$.

□

5) [10 points] Let $S = \{1, x, x^2\}$ and $S' = \{1+x, 1+x^2, x+x^2\}$. [Both are bases of P_2.] Give the transition matrix from S to S'.

Solution. We have:

$$\begin{aligned}
1 &= \frac{1}{2}(1+x) + \frac{1}{2}(1+x^2) - \frac{1}{2}(x+x^2) \\
x &= \frac{1}{2}(1+x) - \frac{1}{2}(1+x^2) + \frac{1}{2}(x+x^2) \\
x^2 &= -\frac{1}{2}(1+x) + \frac{1}{2}(1+x^2) + \frac{1}{2}(x+x^2),
\end{aligned}$$

i.e.,

$$\begin{aligned}
(1)_S &= (1/2, 1/2, -1/2) \\
(x)_S &= (1/2, -1/2, 1/2) \\
(x^2)_S &= (-1/2, 1/2, 1/2).
\end{aligned}$$

So, the transition matrix is:

$$\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

$\square$

6) [15 points] Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and bases for the eigenspaces of A and decide if A diagonalizable?

Solution. We have:

$$\det(xI - A) = \begin{vmatrix} x-2 & -1 & -3 \\ 0 & x-2 & -1 \\ 0 & 0 & x-1 \end{vmatrix} = (x-2)^2(x-1).$$

Hence the eigenvalues are 2 and 1.

The eigenspace associated to 2 is the nullspace of

$$\begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis is $\{(1, 0, 0)\}$.

The eigenspace associated to 1 is the nullspace of

$$\begin{bmatrix} -1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis is $\{(-2, -1, 1)\}$.

Since we only have two vectors in the bases, the matrix is not diagonalizable.

$\square$

7) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & -2 & 1 \\ -2 & 1 & -3 & 0 & 0 & -3 \\ -4 & 2 & -6 & 1 & 3 & -6 \\ -1 & 2 & 0 & -2 & -3 & -3 \end{bmatrix}.$$

Then we have:

$$A \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^{\mathrm{T}} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

You do *not* need to justify any of the items below.

(a) [5 points] Give the rank of A and the dimensions of the row space and of the column space of A?

Solution. Those are all equal to the number of leading ones in the reduced echelon form of A, and hence are equal to 3. $\square$

(b) [5 points] Find a basis for the row space of A made by rows of A.

Solution. Using the echelon form of A^{T}, we see that the first three rows make a basis of the row space. $\square$

(c) [5 points] For each row of A not in the basis of the previous item, give its coordinates with respect to the basis you found.

Solution. Let S be the basis we found [i.e., the first three rows]. Using the echelon form of A^{T} again, we see that:

$$((-1, 2, 0, -2, -3, -3))_S = (3, 0, 1).$$

□

(d) [5 points] Which vectors from the standard basis of $\mathbb{R}^6$ you can add to the vectors in the basis of the row space you found above to obtain a basis of all of $\mathbb{R}^6$?

Solution. Using the echelon form of A itself, we see that we must add $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 0, 1, 0)$, and $(0, 0, 0, 0, 0, 1)$. □