1) Given $f(x)$, compute the derivatives $f'(x)$.

(a) [6 points] $f(x) = \left(\dfrac{e^{2x}}{x^2+1}\right)^5$

Solution.

$$\begin{aligned} f'(x) &= 5\left(\frac{e^{2x}}{x^2+1}\right)^4 \cdot \frac{d}{dx}\left(\frac{e^{2x}}{x^2+1}\right) \\ &= 5\left(\frac{e^{2x}}{x^2+1}\right)^4 \cdot \left(\frac{e^{2x}\cdot 2\cdot (x^2+1) - e^{2x}\cdot 2x}{(x^2+1)^2}\right) \end{aligned}$$

□

(b) [7 points] $f(x) = \cos(2^x)\cdot \arctan(\sqrt{x})$

Solution.

$$f'(x) = -\sin(2^x)\cdot 2^x \cdot \ln(2)\cdot \arctan(\sqrt{x}) + \cos(2^x)\cdot \frac{1}{1+x}\cdot \frac{1}{2\sqrt{x}}.$$

□

(c) [7 points] $f(x) = x^{\ln(x)}$

Solution. Let $g(x) = \ln(f(x)) = \ln(x)\cdot\ln(x) = (\ln(x))^2$. Then,

$$g'(x) = 2\ln(x)\cdot\frac{1}{x}.$$

On the other hand,

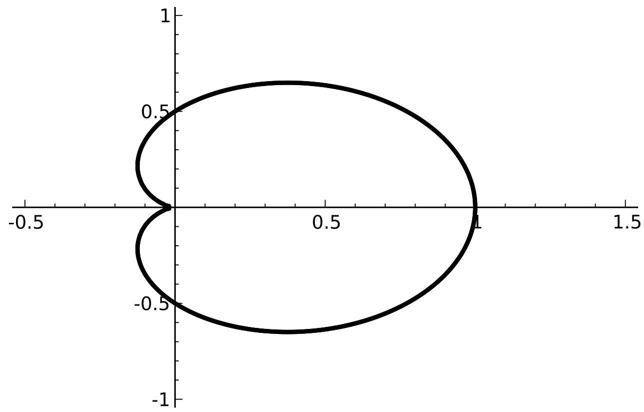
$$g'(x) = \frac{1}{f(x)}\cdot f'(x),$$

and so

$$f'(x) = f(x)g'(x) = x^{\ln(x)}\cdot 2\ln(x)\cdot\frac{1}{x}.$$

□

2) [20 points] The equation $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ gives a *cardioid*. [See the picture below.] Find equation of the tangent line at $(0, 1/2)$.



Solution. Taking derivatives of both sides of the equation, we get

$$2x + 2yy' = 2(2x^2 + 2y^2 - x) \cdot (4x + 4yy' - 1).$$

If $x = 0$ and $y = 1/2$, we get

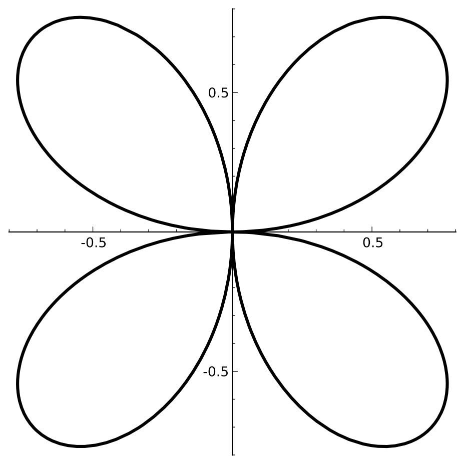
$$y' = 1 \cdot (2y' - 1).$$

Thus, at $(0, 1/2)$ we have that $y' = 1$ and the equation of the tangent line is $y - 1/2 = 1 \cdot (x - 0)$, or $y = x + 1/2$. □

3) [20 points] Consider the parametrized curve [pictured below] given by

$$x = \cos(t)\ \sin(2t),$$
$$y = \sin(t)\ \sin(2t).$$

Show that the tangent lines for $t = \pi/4$ and $t = -\pi/4$ are perpendicular.



Solution. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sin(t)\sin(2t) + 2\cos(t)\cos(2t)}{\cos(t)\sin(2t) + 2\sin(t)\cos(2t)}$$

So, at $t = \pi/4$ and $t = -\pi/4$ we get slopes

$$y' = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1 \qquad \text{and} \qquad y' = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

Since the slopes multiply to -1, the lines are perpendicular.

□

4) [20 points] The circumference of a sphere [i.e., the length of the "equator" of the sphere] was measured to be 8 cm, with possible error of 0.5 cm. Estimate the maximal error *and* the relative error that can occur with both surface area and volume of the sphere.

[**Hint:** If r is the radius of a sphere, its circumference C, surface area A, and volume V are given by $C = 2\pi r$, $A = 4\pi r^2$, and $V = 4\pi r^3/3$ respectively. *You might need to write A and V in terms of C [instead of r].*]

Solution. We have $C = 2\pi r$, and hence $r = C/(2\pi)$. Then, $A = 4\pi(C/(2\pi))^2 = C^2/\pi$, and $V = 4\pi(C/(2\pi))^3/3 = C^3/(6\pi^2)$. I.e.,

$$A = \frac{C^2}{\pi} \qquad \text{and} \qquad V = \frac{C^3}{6\pi^2}.$$

So,

$$\mathrm{d}A = \frac{2C}{\pi}\mathrm{d}C \qquad \text{and} \qquad \mathrm{d}V = \frac{3C^2}{6\pi^2}\mathrm{d}C = \frac{C^2}{2\pi^2}\mathrm{d}C.$$

So, if $C = 8$ and $\Delta C = 0.5$, we get maximal errors

$$\Delta A \approx 2 \cdot \frac{8}{\pi} \cdot 0.5 = \frac{8}{\pi} \qquad \text{and} \qquad \Delta V \approx \frac{8^2}{2\pi^2} \cdot 0.5 = \frac{16}{\pi^2}.$$

Now, for $C = 8$ we get $A = 64/\pi$ and $V = 256/(3\pi^2)$. Then, the relative errors when $C = 8$ are

$$\frac{\Delta A}{A} \approx \frac{8/\pi}{64/\pi} = \frac{1}{8} \qquad \text{and} \qquad \frac{\Delta V}{V} \approx \frac{16/\pi^2}{256/3\pi^2} = \frac{3}{16}.$$

□

5) [20 points] A 10 ft long ladder rests against a [vertical] wall. If the bottom of the ladder slides away from wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution. This is Example 2 on pg. 264 from the text. □