1) Compute the following limits. If they do not exist or are infinite, check the side limits.

(a) [5 points] $\lim_{x\to 3} \frac{x^3 - 9x}{x - 3}$

Solution.

$$\begin{aligned}\lim_{x\to 3} \frac{x^3 - 9x}{x - 3} &= \lim_{x\to 3} \frac{x(x^2 - 9)}{x - 3} \\ &= \lim_{x\to 3} \frac{x(x-3)(x+3)}{x - 3} \\ &= \lim_{x\to 3} x(x+3) = 18.\end{aligned}$$

□

(b) [5 points] $\lim_{x\to 1} \frac{\left(\frac{1}{1-x}\right)}{x - 1}$

Solution.

$$\begin{aligned}\lim_{x\to 1} \frac{\left(\frac{1}{1-x}\right)}{x - 1} &= \lim_{x\to 1} \frac{1}{1-x} \cdot \frac{1}{x-1} \\ &= \lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty.\end{aligned}$$

[Note, for the last step, that $\frac{-1}{(x-1)^2}$ is always negative!] □

(c) [5 points] $\lim_{x \to -\infty} \frac{2x^2 - \sqrt{x}}{x^2 + 3x^{-1}}$

Solution.

$$\begin{aligned}
\lim_{x \to -\infty} \frac{2x^2 - \sqrt{x}}{x^2 + 3x^{-1}} &= \lim_{x \to -\infty} \frac{x^2(2 - x^{-3/2})}{x^2(1 + 3x^{-3})} \\
&= \lim_{x \to -\infty} \frac{2 - x^{-3/2}}{1 + 3x^{-3}} = 2.
\end{aligned}$$

□

(d) [10 points] $\lim_{x \to 0} \frac{\tan(4x)}{9x}$

Solution.

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan(4x)}{9x} &= \lim_{x \to 0} \frac{\sin(4x)/\cos(4x)}{9x} \\
&= \lim_{x \to 0} \frac{1}{\cos(4x)} \frac{\sin(4x)}{9x} \\
&= \left(\lim_{x \to 0} \frac{1}{\cos(4x)} \right) \cdot \left(\lim_{x \to 0} \frac{\sin(4x)}{9x} \right) \\
&= \frac{1}{\cos(0)} \cdot \left(\lim_{y \to 0} \frac{\sin(y)}{9y/4} \right) \qquad \text{[make substitution } y = 4x\text{]} \\
&= 1 \cdot \left(\frac{4}{9} \cdot \lim_{y \to 0} \frac{\sin(y)}{y} \right) \\
&= \frac{4}{9}.
\end{aligned}$$

□

2) [10 points] Compute $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x^3}\cdot \mathrm{e}^x)$.

Solution. We have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x^3}\cdot \mathrm{e}^x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x^3})\cdot \mathrm{e}^x + \sqrt{x^3}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{3/2})\cdot \mathrm{e}^x + \sqrt{x^3}\cdot \mathrm{e}^x\\ &= \frac{3}{2}x^{1/2}\cdot \mathrm{e}^x + \sqrt{x^3}\cdot \mathrm{e}^x.\end{aligned}$$

□

3) [15 points] Find the equation of the line tangent to the graph of $f(x) = \dfrac{x+1}{x^2+1}$ at $x = 0$.

Solution. We have:

$$\begin{aligned}f'(x) &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x+1)\cdot(x^2+1) - (x+1)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)}{(x^2+1)^2}\\ &= \frac{1\cdot(x^2+1)-(x+1)\cdot 2x}{(x^2+1)^2}\\ &= \frac{-x^2-2x+1}{(x^2+1)^2}.\end{aligned}$$

So, $f'(0) = 1$.

The equation of the tangent line at $x = 0$ is given by $y - f(0) = f'(0)(x-0)$, i.e., $y - 1 = x$ [or $y = x + 1$].

□

4) [10 points] Let

$$f(x) = \begin{cases} x^3 + 1, & \text{if } x < 0, \\ 1 - x^3, & \text{if } x \geq 0. \end{cases}$$

Is $f(x)$ continuous at $x = 0$?

Solution. We have

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} 1 - x^3 = 1,$$

and

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} x^3 + 1 = 1,$$

Thus, $\lim_{x \to 0} f(x) = 1 = f(0)$, and therefore the function is continuous at $x = 0$. □

5) [10 points] Let $f(x) = \sin(x^2) - x$. Express $f'(x)$ as a limit. [Do *not* compute the limit!]

Solution. We have:

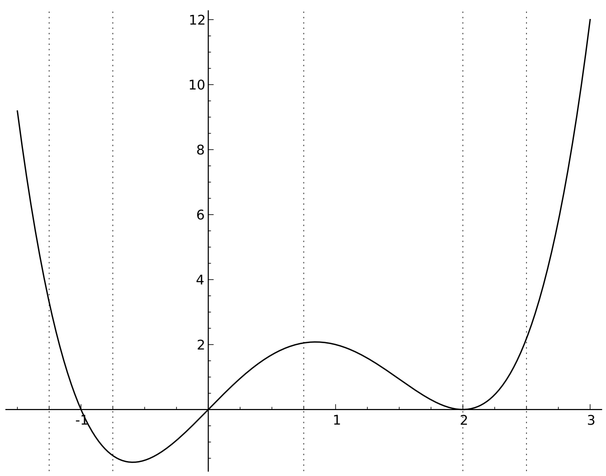
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin((x+h)^2) - (x+h) - (\sin(x^2) - x)}{h}.$$

□

6) [15 points] Show that the graphs of $f(x) = x\mathrm{e}^x$ and $g(x) = \cos(x)$ intersect and specify an interval in the x-axis for which we have at least one intersection.
[**Hint:** Use the *Intermediate Value Theorem.*]

Solution. There is an intersection if $x\mathrm{e}^x = \cos(x)$ has a solution, in other words, if $x\mathrm{e}^x - \cos(x) = 0$ has a solution. So, let $h(x) = x\mathrm{e}^x - \cos(x)$. Then, $h(0) = -1 < 0$. Also, we have that $h(1) > 0$, since $|\cos(1)| \leq 1$ and $1 \cdot \mathrm{e}^1 = \mathrm{e} > 2$. Thus, there is a solution in $[0, 1]$, and this solution gives us an x-value of the intersection of the two graphs. □

7) [20 points] The graph of $f(x)$ is given below.



Put the following numbers in non-decreasing order: $f'(-1.25)$, $f'(-0.75)$, $f'(0.75)$, $f'(2)$, $f'(2.5)$. [You do *not* need to show work on this one.]
[**Note:** To put -1, -2, 0, 3, 0, 1.25 in non-decreasing order, is to put them in the order: -2, -1, 0, 0, 1.25, 3. In other words, it is in increasing order, except consecutive number might be equal.]

Solution. We just need to check the slopes of the tangent line at the given points. In particular, $f'(-1.25)$ and $f'(-0.75)$ are negative, $f'(2) = 0$, and $f'(0.75)$ and $f'(2.5)$ are positive. So, the order is: $f'(-1.25)$, $f'(-0.75)$, $f'(2)$, $f'(0.75)$, $f'(2.5)$. □