1) [15 points] If $f(x) = \ln(\cos(x^2))$, compute $f''(x)$.

Solution. We have:

$$f'(x) = \frac{1}{\cos(x^2)} \cdot (-\sin(x^2)) \cdot 2x = -2x \cdot \tan(x^2).$$

Thus,

$$f''(x) = -2 \cdot \tan(x^2) - 2x \cdot \sec^2(x^2) \cdot 2x = -2 \cdot \tan(x^2) - 4x^2 \cdot \sec^2(x^2).$$

□

2) [15 points] Find an approximation for arctan(1.1). What is the percentage error in this case?

[**Note:** Remember that I use $\arctan(x)$ for what the book denotes by $\tan^{-1}(x)$. In other words, the book would ask for approximation of $\tan^{-1}(1.1)$.]

Solution. We have that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x)) = \frac{1}{x^2+1}.$$

Thus,

$$\arctan(1.1) \approx \arctan(1) + \frac{1}{1^2+1} \cdot 0.1 = \pi/4 + 0.05.$$

The percentage error is:

$$\left| \frac{\arctan(1.1) - (\pi/4 + 0.05)}{\arctan(1.1)} \right|.$$

[You could not do this without a calculator, but this number is approximately 0.0029, or 0.29%.] □

3) [15 points] Find the equation of the line tangent to the curve $x^2 + \sin(y) = xy^2 + 1$ at the point $(1, 0)$.

Solution. Using implicit differentiation, we have:

$$2x + \cos(y) \cdot y' = y^2 + 2xy \cdot y'.$$

At the point $(1, 0)$, this gives us

$$2 + y' = 0$$

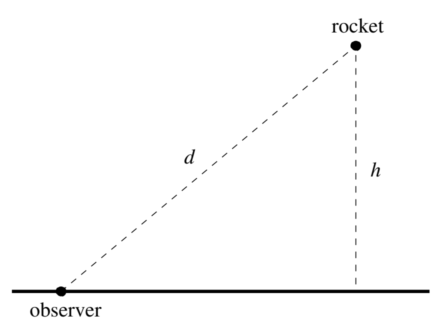
So, $y' = -2$. Hence, the equation of the tangent line is:

$$(y - 0) = -2(x - 1) \qquad \text{or} \qquad y = -2x + 2.$$

□

4) [15 points] A rocket is takes off (vertically) with initial speed of 300 miles per hour. How fast is the distance between an observer (on the ground) 3 miles away from the take off point and the rocket is increasing when the rocket is 4 miles high?

[**Note:** The values of this problem were chosen for simplicity in computations, not for accuracy.]



Solution. If h is the height of the rocket and d is the distance to the observer, we have that Pythagoras gives us

$$3^2 + h^2 = d^2.$$

Taking derivatives with respect to time, we get:

$$2h \cdot h' = 2d \cdot d' \qquad \text{or} \qquad 300h = d \cdot d'.$$

When $h = 4$, the formula above gives $d = 5$. So,

$$d' = \frac{300 \cdot 4}{5} = 240.$$

So, the distance is increasing 240 miles per hour. □

5) Let $f(x) = x^3 - 12x + 1$.

(a) [10 points] Find where $f(x)$ is increasing and where it is decreasing.

Solution. We have $f'(x) = 3x^2 - 12$. So, $f'(x) = 0$ at $x = \pm 2$. By plugging in $x = -3, 0, 3$ in $f'(x)$ we get that:

- $f'(x)$ is positive, and hence $f(x)$ is increasing, in $(-\infty, -2)$ and $(2, \infty)$;
- $f'(x)$ is negative, and hence $f(x)$ is decreasing, in $(-2, 2)$.

□

(b) [10 points] Find where $f(x)$ is concave up and where it is concave down.

Solution. We have $f''(x) = 6x$. Thus:

- $f''(x)$ is positive, and hence $f(x)$ is concave up, in $(0, \infty)$;
- $f''(x)$ is negative, and hence $f(x)$ is concave down, in $(-\infty, 0)$.

□

(c) [10 points] Find the x-coordinate of all inflection points and local maxima and minima of $f(x)$.

Solution. From item (a), we see that we have local maximum when $x = -2$ and a local minimum when $x = 2$. From (b) we see that we have an inflection point at $x = 0$. □

(d) [10 points] Find the *global* maximum and minimum of $f(x)$ in the interval $[-3, 5]$, indicating for which value(s) of x they occur.

Solution. We have local maximum and minimum at $x = 2$ and $x = -2$. We have $f(2) = -15$ and $f(-2) = 17$. At the end points, we get $f(-3) = 10$ and $f(5) = 66$. □