1) [10 points] Let $f(x) = \dfrac{(x-1)^2(x+2)^2}{x^2-3x+2}$.

(a) Give the domain of $f(x)$.

(b) Give the values of x for which we have $f(x) = 0$.

(c) Give the *intervals* where $f(x) < 0$.

Solution. For (a) we only need to see what are the zeros in the denominators. Solving $x^2 - 3x + 2 = 0$, we obtain $x = 1$ and $x = 2$. So, the domain all real numbers different from 1 and 2.

For (b) we need to see what are the values of x *in the domain* that make the numerator zero. The zeros of the numerator are $x = 1$ and $x = -2$. So, the only value of x that make $f(x) = 0$ is $x = -2$.

For (c) one can mark the points in which $f(x)$ is not defined and where it is zero [from parts (a) and (b)] and check the signs for choices in between them. We get that $f(x) > 0$ in $(-\infty, -2)$, $(-2, 1)$, and $(2, \infty)$, and $f(x) < 0$ for in $(1, 2)$. □

2) Compute the following limits.

(a) [5 points] $\displaystyle\lim_{x\to 1} \frac{x^3 + e^x}{x^2 - 2x + 1}$

Solution. Since plugging $x = 1$ gives us "$(1 + e)/0$" [and $(1 + e) \neq 0$], we have that it is some kind of infinite limit. Since the numerator and denominator are positive close to 1 [on either side], we get that the limit is $+\infty$. □

(b) [5 points] $\displaystyle\lim_{x\to -\infty} \frac{2x^3 - 3x + 1}{x^3 + \sqrt[3]{x}}$

Solution. Since the highest powers of x in the numerator and denominator are equal, we get that the limit is the quotient of their coefficients. Thus the limit is $2/1 = 2$. □

(c) [5 points] $\lim_{x\to 0} \dfrac{\cos(2x^2) - e^x}{3\tan(3x)}$

Solution. We use L'Hospital's rule:

$$\begin{aligned}\lim_{x\to 0} \frac{\cos(2x^2) - e^x}{3\tan(3x)} &= \lim_{x\to 0} \frac{-\sin(2x^2)\cdot 4x - e^x}{3\sec^2(3x)\cdot 3} && \text{[L'Hospital's Rule]}\\ &= \lim_{x\to 0} \frac{-(4x\sin(2x^2) + e^x)}{9\sec^2(3x)}\\ &= -\frac{1}{9}.\end{aligned}$$

□

3) Compute the following derivatives:

(a) [5 points] $\dfrac{d}{dx}(\arctan(2^x))$

Solution.

$$\frac{d}{dx}(\arctan(2^x)) = \frac{1}{1+(2^x)^2}\cdot 2^x \cdot \ln(2).$$

□

(b) [5 points] $\dfrac{d}{dx}(\ln(x)\sqrt{x})$

Solution.

$$\frac{d}{dx}\left(\ln(x)\sqrt{x}\right) = \frac{1}{x}\cdot\sqrt{x} + \ln(x)\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}}$$

□

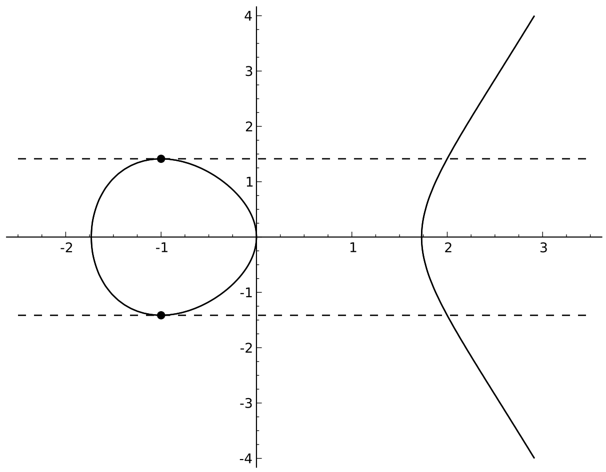
(c) [5 points] $\dfrac{d}{dx}\left(\frac{f(g(x))}{f(x)g(x)}\right)$ [your answer should be a formula in terms of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$]

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(g(x))}{f(x)g(x)}\right) &= \frac{\frac{d}{dx}(f(g(x)))\cdot(f(x)g(x)) - f(g(x))\frac{d}{dx}(f(x)g(x))}{(f(x)g(x))^2}\\ &= \frac{(f'(g(x))\,g'(x))f(x)\,g(x) - f(g(x))(f'(x)g(x) + f(x)g'(x))}{(f(x)g(x))^2}\end{aligned}$$

□

4) [10 points] Find the coordinates [x and y-coordinates] of the points with horizontal tangent line on the curve given by $y^2 = x^3 - 3x$.
[**Note:** The graph is given below, but you *cannot* use it in the solution. It might be useful to verify if your answer seems correct, though.]



Solution. We have

$$2yy' = 3x^2 - 3 \qquad \text{and so} \qquad y' = \frac{3(x^2-1)}{2y}.$$

So, solving $y' = 0$ we obtain $x = \pm 1$. So,

$$y^2 = (\pm 1)^3 - 3\,(\pm 1) = \mp 2.$$

Since this is non-negative [as it is equal to y^2], we must discard the negative value, which is given by $x = 1$.

For $x = -1$ we get $y = \pm\sqrt{2}$. So, there are two points with horizontal tangent line: $(-1, \sqrt{2})$ and $(-1, -\sqrt{2})$.

□

5) [10 points] Find the maximum and minimum of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 6]$ as well as the values of x in which they occur.

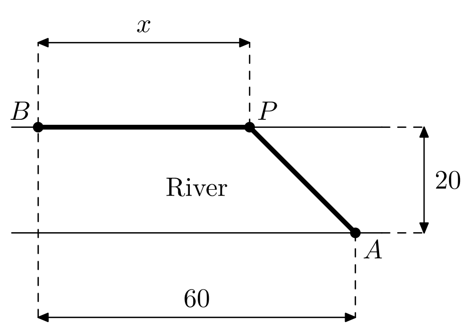
Solution. We have $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 4)(x - 1)$. So, the critical points are $x = 1$ and $x = 4$.

We have $f(0) = 7$, $f(1) = 18$, $f(4) = -9$ and $f(6) = 43$. So, the minimum is -9, and occurs at $x = 4$, and the maximum is 43, and occurs at $x = 6$. □

6) [10 points] [In this question you will set up, *but not solve* an optimization problem.]

A person can swim at speed of 3 feet per second and run at a speed of 5 feet per second. He has to cross a 20 feet long river from point A [see picture below] to get to point B which is 60 feet to the left of the opposite margin of point A.

Find a function that gives the *time* it will take the person to go from A to B if he swims first to a point x feet to the right of B [labeled P in the picture] and then runs the rest of the way. [This function should involve x only!] Also, give a closed interval for the values of x in which the minimum of this function would give us the minimum time to go from A to B.

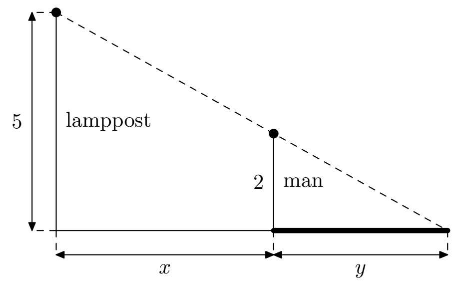


Solution. The function is the time to swim from A to P and then walk from P to B, so it is the distance from A to P, divided by how fast the person can swim, plus the distance from P to B, divided by how fast the person can walk:

$$f(x) = \frac{\sqrt{20^2 + (60 - x)^2}}{3} + \frac{x}{5}.$$

The range for x is $[0, 60]$. □

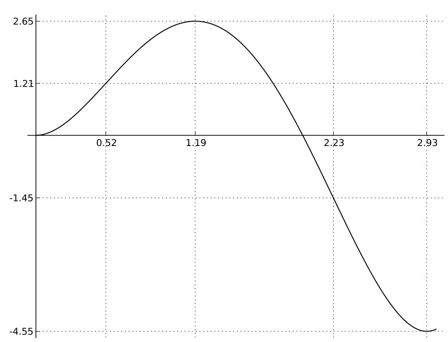
7) [10 points] A 2 meter tall man runs away from a 5 meter high lamppost at a speed of 3 meter per second. [See picture below.] How fast is the length of the shadow [denote by y in the picture] increasing?



Solution. Using similar triangles we have $\frac{2}{y} = \frac{5}{x+y}$, and so, $y = \frac{2x}{3}$.

Taking derivatives, we have $y' = \frac{2x'}{3} = 2$. So, the shadow increases at rate of 2 meters per second. □

8) [10 points] The *position* of a particle moving along a straight line at time t is given by $s(t)$. The graph of $s(t)$ for t in $[0, 3]$ is given below. [Coordinates of all local maxima/minima and inflexion points are given in the graph. Note that the tangent line at $t = 0$ is *horizontal*!]



Answer the following based on the graph. [No need to justify these.]

(a) At what time(s) is the *velocity* [not position!] maximal and at what time(s) is it minimal?

(b) At what time(s) was the velocity equal to zero?

(c) When was the *acceleration* [not position, nor velocity!] negative? Give your answer as an interval.

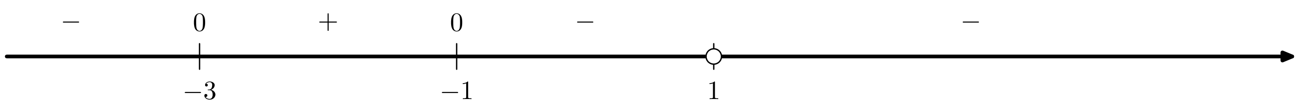
Solution. The velocity is maximal when the tangent line is the steepest up, and so at $t = 0.52$. It is minimal when the slope of the tangent line is the steepest down, and so at $t = 2.23$.

The velocity is zero when the tangent line is horizontal, so at $t = 0$, $t = 1.19$ and $t = 2.93$.

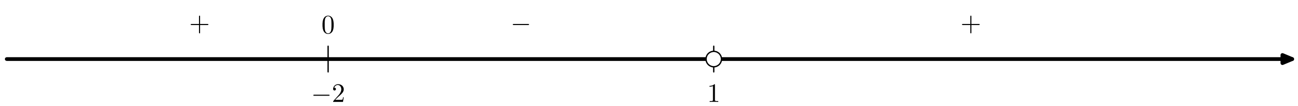
The acceleration is negative when the graph is concave down, so in $(0.52, 2.23)$. □

9) [10 points] Sketch the graph of a function $f(x)$ which satisfies all of the following conditions [draw concavities carefully!]:

- domain is all real numbers except 1;
- x-intercepts are -3.5, -2.5, 0, and y-intercept is 0;
- $f(-3) = -1$, $f(-1) = 2$;
- $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 1_-} f(x) = -\infty$, $\lim_{x\to 1_+} f(x) = \infty$;
- the sign of the derivative is given by:



- the sign of the second derivative is given by:



Solution. Needs to be drawn by hand... □