Final

M552 – Modern Algebra II

May 2nd, 2012

We assume that R is a commutative ring with $1 \neq 0$.

1. Let R be a domain with field of fractions F and M be an R-module. Show that if $\operatorname{rank}(M) = r$, then $\dim_F(F \otimes_R M) = r$.

2. Let F be a field and M be a finitely generated $F[x]$-module. Show that M is projective if, and only if, M is isomorphic [as $F[x]$-module] to $F[x] \otimes V$ for some finite dimensional vector F-space V.

3. Let $q = p^n$, where p is an odd prime, and consider $f = x^q - x - 1 \in \mathbb{F}_q[x]$. Show that every irreducible factor of f has degree p. [**Hint:** if α is a root, then show that $\alpha^{(q^p)} = \alpha$.]

4. Let $F \subseteq K \subseteq L$ be fields, with K/F Galois, $\alpha \in L$ such that $F[\alpha]/F$ is also Galois. Assume also that $\operatorname{Gal}(K/F) \cong A_7$ and $\operatorname{Gal}(F[\alpha]/F) \cong Z_4 \times Z_7$. Find $\operatorname{Aut}(K[\alpha]/K)$.