1. [50 points] An R-module is called *artinian* if it satisfies the descending chain condition for submodules.

Suppose L, M and N are R-modules yielding the short exact sequence:

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

Show that if L and N are artinian, then so is M.

[**Note:** The converse is also true and easier to prove.]

Proof. Let $M_1 \subseteq M_2 \subseteq \cdots$ be a sequence of submodules on M.

Since L is artinian, ψ is injective [and thus an isomorphism onto $\psi(L)$], we have that $M_1 \cap \psi(L) \subseteq M_2 \cap \psi(L) \subseteq \cdots$ is stationary [as its preimage is], i.e., there exists l such that $M_i \cap \psi(L) = M_l \cap \psi(L)$ for all $i \geq l$.

Since N is artinian and $\phi(M_i)$ is a submodule of N, we have that $\phi(M_1) \subseteq \phi(M_2) \subseteq \cdots$ is also stationary, i.e., there exists n such that $\phi(M_i) = \phi(M_n)$ for all $i \geq n$.

Let $m = \max\{l, n\}$. Then, $M_i = M_n$ for $i \geq n$. Indeed: let $x \in M_m$. [We need to show that $x \in M_i$ for all $i \geq m$.] We have that $\phi(x) \in \phi(M_m) = \phi(M_i)$. Thus, there is $y \in M_i$ such that $(x - y) \in \ker(\phi) = \psi(M)$, so $x - y \in M_m \cap \psi(L)$ [as $y \in M_i \subseteq M_m$], and hence $x - y \in M_i \supseteq M_i \cap \psi(L) = M_m \cap \psi(L)$. Thus, $x = y + (x - y) \in M_i$.

□

2. [50 points] Let M and N be R-modules and M' and N' be submodules of M and N respectively. Define L as the sumbodule of $M \otimes_R N$ generated by the set $\{x \otimes y \in M \otimes_R N \ : \text{ either } x \in M' \text{ or } y \in N'\}$. Show that $M/M' \otimes_R N/N' \cong (M \otimes_R N)/L$.

[**Note:** If the proof is straightforward, you can just say that a map is bilinear without proof.]

Proof. Consider the map $\phi : M \times N \to M/M' \otimes N/N'$ defined by $\phi(m, n) = (m + M') \otimes (n + N')$. This is clearly bilinear, and hence induces a homomorphism $\Phi : M \otimes N \to M/M' \otimes N/N'$ such that $\Phi(m \otimes n) = (m + M') \otimes (n + N')$.

Note that $L \subseteq \ker(\Phi)$, as if either $m \in M'$ or $n \in N'$, then $\Phi(m \otimes n) = 0$. Thus, we have a naturally defined homomorphism $\tilde{\Phi} : (M \otimes N)/L \to M/M' \otimes N/N'$, with $\tilde{\Phi}(m \otimes n + L) = (m + M') \otimes (n + N')$.

Now, consider the map $\psi : M/M' \times N/N' \to (M \otimes N)/L$, defined by $\psi(m + M', n + N') = m \otimes n + L$. This is well defined, as if $m' - m \in M'$ and $n' - n \in N'$, then

$$\begin{aligned} m' \otimes n' + L &= (m + (m' - m)) \otimes (n + (n' - n)) + L \\ &= m \otimes n + (m \otimes (n' - n) + (m' - m) \otimes n + (m' - m) \otimes (n' - n)) + L \\ &= m \otimes n + L. \end{aligned}$$

Thus, we have a homomorphism $\Psi : M/M' \otimes N/N' \to (M \otimes N)/L$, such that $\Psi((m + M') \otimes (n + N')) = m \otimes n + L$.

Clearly, $\tilde{\Phi}$ and Ψ are inverses of each other.

□