FIELD THEORY

MATH 552

Contents

1. Algebraic Extensions

1.1. **Finite and Algebraic Extensions.**

Definition 1.1.1. Let 1_F be the multiplicative unity of the field F.

(1) If $\sum_{i=1}^{n} 1_F \neq 0$ for any positive integer n, we say that F has *characteristic* 0.
(2) Otherwise, if p is the smallest positive integer such that $\sum_{i=1}^{p} 1_F = 0$, then F has *characteristic* p. (In this case, p is necessarily prime.)
(3) We denote the characteristic of the field by $\mathrm{char}(F)$.

(4) The *prime field* of F is the smallest subfield of F. (Thus, if $\mathrm{char}(F) = p > 0$, then the prime field of F is $\mathbb{F}_p \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$ (the filed with p elements) and if $\mathrm{char}(F) = 0$, then the prime field of F is $\mathbb{Q}$.)

(5) If F and K are fields with $F \subseteq K$, we say that K is an *extension* of F and we write K/F. F is called the *base field*.

(6) The *degree* of K/F, denoted by $[K : F] \stackrel{\text{def}}{=} \dim_F K$, i.e., the dimension of K as a vector space over F. We say that K/F is a *finite extension* (resp., *infinite extension*) if the degree is finite (resp., infinite).

(7) α is *algebraic* over F if there exists a polynomial $f \in F[X] - \{0\}$ such that $f(\alpha) = 0$.

Definition 1.1.2. If F is a field, then

$$F(\alpha) \stackrel{\text{def}}{=} \left\{ \frac{f(\alpha)}{g(\alpha)} \ : \ f, g \in F[X] \text{ and } g(\alpha) \neq 0 \right\},$$

is the smallest extension of F containing α. (Hence α is algebraic over F if, and only if, $F[\alpha] = F(\alpha)$.)

In the same way,

$$\begin{aligned} F(\alpha_1, \dots, \alpha_n) &\stackrel{\text{def}}{=} \left\{ \frac{f(\alpha_1, \dots, \alpha_n)}{g(\alpha_1, \dots, \alpha_n)} \ : \ f, g \in F[X_1, \dots, X_n] \text{ and } g(\alpha_1, \dots, \alpha_n) \neq 0 \right\} \\ &= F(\alpha_1, \dots, \alpha_{n-1})(\alpha_n) \end{aligned}$$

is the smallest extension of F containing $\{\alpha_1, \dots, \alpha_n\}$.

Definition 1.1.3. If K/F is a finite extension and $K = F[\alpha]$, then α is called a *primitive element* of K/F.

Proposition 1.1.4. *For any $f \in F[X] - \{0\}$ there exists an extension K/F such that f has a root in K. (E.g., $K \stackrel{\text{def}}{=} F[X]/(g)$, where g is an irreducible factor of f.)*

Theorem 1.1.5. *If $p(X) \in F[X]$ is irreducible of degree n, $K \stackrel{\text{def}}{=} F[X]/(p(X))$ and θ is the class of X in K, then θ is a root of $p(X)$ in K, $[K : F] = n$ and $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ is an F-basis of K.*

Remark 1.1.6. Observe that $F[\theta]$ (polynomials over F evaluated at θ), where θ is a root of an irreducible polynomial $p(X)$, is then a *field*. Observe that $1/\theta$ can be obtained with the *extended Euclidean algorithm*: if $d(X)$ is the $\gcd(X, p(X))$ and $d(X) = a(X) \cdot X + b(X) \cdot p(X)$, the $1/\theta = a(\theta)$.

Definition 1.1.7. If α is algebraic over F, then there is a *unique* monic irreducible over F that has α as a root, called the *irreducible polynomial* (or *minimal polynomial*) of α over F, and we shall denote it $\min_{\alpha,F}(X)$. [**Note:** $(\min_{\alpha,F}(X)) = \ker\phi$, where $\phi : F[X] \to F[\alpha]$ is the evaluation map.]

Corollary 1.1.8. *If α is algebraic over F, then $F(\alpha) = F[\alpha] \cong F[x]/(\min_{\alpha,F})$, and $[F[\alpha] : F] = \deg \min_{\alpha,F}$.*

Proposition 1.1.9. *If K is a finite extension of F and α is algebraic over K, then α is algebraic over F and $\min_{\alpha,K}(X) \mid \min_{\alpha,F}(X)$.*

Definition 1.1.10. Let $\phi : R \to S$ be a ring homomorphism. If $f(X) = a_n X^n + \cdots + a_1 X + a_0$, then $f^\phi \stackrel{\text{def}}{=} \phi(a_n)X^n + \cdots + \phi(a_1)X + \phi(a_0) \in S[X]$. [Note that $f \mapsto f^\phi$ is a ring homomorphism.]

Theorem 1.1.11. *Let $\phi : F \to F'$ be an isomorphism, and $f \in F[X]$ be an irreducible polynomial. If α is a root of f in some extension of F and α' is a root of f^ϕ in some extension of F', then there exists an isomorphism $\Phi : F[\alpha] \to F'[\alpha']$ such that $\Phi(\alpha) = \alpha'$ and $\Phi|_F = \phi$.*

Definition 1.1.12. K/F is an *algebraic extension* if every $\alpha \in K$ is algebraic over F.

Proposition 1.1.13. *If $[K : F] < \infty$, then K/F is algebraic.*

Remark 1.1.14. The converse is false. E.g., $\bar{\mathbb{Q}} \stackrel{\text{def}}{=} \{\alpha \in \mathbb{C} \,:\, \alpha \text{ is algebraic over } \mathbb{Q}\}$ is an infinite algebraic extension of $\mathbb{Q}$.

Proposition 1.1.15. *If L is a finite extension K and K is a finite extension of F, then*

$$[L : F] = [L : K] \cdot [K : F].$$

Moreover, if $\{\alpha_1, \dots, \alpha_n\}$ is an F-basis of K and $\{\beta_1, \dots, \beta_m\}$ is a K-basis of L, then $\{\alpha_i \cdot \beta_j : i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}\}$ is an F-basis of L.

Definition 1.1.16. $\{\alpha_1, \dots, \alpha_n\}$ *generates* K/F if $K = F(\alpha_1, \dots, \alpha_n)$ and K/F is *finitely generated*. (Not necessarily algebraic!)

Proposition 1.1.17. $[K : F] < \infty$ *if, and only if, K is finitely generated over F by algebraic elements.*

Corollary 1.1.18. *Let K/F be an arbitrary extension, then*

$$E \overset{\text{def}}{=} \{\alpha \in K \;:\; \alpha \textit{ is a algebraic over } F\},$$

is a subfield of K containing F.

Definition 1.1.19. If F and K are fields contained in the field $\mathcal{F}$, then the *composite* (or *compositum*) of F and K is the smallest subfield of $\mathcal{F}$ containing F and K, and is denoted by $F\,K$.

Proposition 1.1.20. (1) *In general, we have:*

$$F\,K = \left\{ \frac{\alpha_1\beta_1 + \cdots + \alpha_m\beta_m}{\gamma_1\delta_1 + \cdots + \gamma_n\delta_n} \;:\; \alpha_i, \gamma_i \in F;\; \beta_j, \delta_j \in K;\; \gamma_1\delta_1 + \cdots + \gamma_n\delta_n \neq 0 \right\}$$

(2) *If K_1/F and K_2/F are finite extensions, with $K_1 = F[\alpha_1, \dots, \alpha_m]$ and $K_2 = F[\beta_1, \dots, \beta_n]$, then $K_1\,K_2 = F[\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n]$, and $[K_1\,K_2 : F] \leq [K_1 : F] \cdot [K_2 : F]$.*

Definition 1.1.21. Let $\mathcal{C}$ be a class of field extensions. We say that $\mathcal{C}$ is *distinguished* if the following three conditions are satisfied:

(1) Let $F \subseteq K \subseteq L$. Then, L/F is in $\mathcal{C}$ if, and only if, L/K and K/F are in $\mathcal{C}$.

(2) If K_1 and K_2 are extensions of F, both contained in $\mathcal{F}$, then if K_1/F is in $\mathcal{C}$, then $K_1 K_2/K_2$ is also in $\mathcal{C}$.
(3) If K_1 and K_2 are extensions of F, both contained in $\mathcal{F}$, then if K_1/F and K_2/F are in $\mathcal{C}$, then $K_1 K_2/F$ is also in $\mathcal{C}$. [Note that this follows from the previous two.]

Definition 1.1.22. Let $\mathcal{C}$ be a class of field extensions. We say that $\mathcal{C}$ is *quasi-distinguished* if the following three conditions are satisfied:

(1′) Let $F \subseteq K \subseteq L$. Then, if L/F is in $\mathcal{C}$ then L/K in $\mathcal{C}$.
(2) Same as (2) of distinguished.
(3) Same as (3) of distinguished.

Remark 1.1.23. The above definition is *not* standard.

Proposition 1.1.24. *The classes of algebraic extensions and finite extensions are distinguished.*

1.2. **Algebraic Closure.**

Definition 1.2.1. Let K and L be extensions of F.

(1) An *embedding* (i.e., an injective homomorphism) $\phi : K \to L$ is *over* F if $\phi|_F = \mathrm{id}_F$.
(2) If E/K and $\psi : E \to L$ is also an embedding, we say that ψ is *over* ϕ, or is an *extension* of ϕ, if $\psi|_K = \phi$.

Remark 1.2.2. Remember that if $\phi : F \to F'$ is field homomorphism, then ϕ is either injective or $\phi \equiv 0$.

Definition 1.2.3. An *algebraic closure* of F is an algebraic extension K in which any polynomial in $F[X]$ *splits* [i.e., can be written as a product of linear factors] in $K[X]$. We say that F is *algebraically closed* if it is an algebraic closure of itself.

Lemma 1.2.4. *Let K/F be algebraic. If $\phi : K \to K$ is an embedding over F, then ϕ is an automorphism.*

Lemma 1.2.5. *Let F and K be subfields of $\mathcal{F}$ and $\phi : \mathcal{F} \to L$ be an embedding into some field L. Then $\phi(F\,K) = \phi(F)\,\phi(K)$.*

Theorem 1.2.6. (1) *For any field F, there exists an algebraic closure of F.*
(2) *An algebraic closure of F is algebraically closed.*

Definition 1.2.7. If

$$f(X) = \sum_{i=0}^{n} a_i\,X^i \in F[X],$$

then the *formal derivative* of f is

$$f'(X) = \sum_{i=0}^{n} i\,a_i\,X^{i-1}.$$

Remark 1.2.8. The same formulas from calculus still hold (product rule, chain rule, etc.).

Lemma 1.2.9. *Let $f \in F[X]$ and α a root of f. Then α is a multiple root if, and only if, $f'(\alpha) = 0$.*

Lemma 1.2.10. *Let $\phi : F \to F'$ be an embedding, $c, a_1, \dots, a_k \in F$, and $f \stackrel{\text{def}}{=} c(X - a_1)\cdots(X - a_k) \in F[X]$. Then, $f^{\phi}(X) = \phi(c)(X - \phi(a_1))\cdots(X - \phi(a_k))$.*

Theorem 1.2.11. *Let $f \in F[X]$ be an irreducible polynomial. If f splits in K as $f = c(X - \alpha_1)^{n_1}\cdots(X - \alpha_k)^{n_k}$, with the α_i's distinct, then $n_1 = \cdots = n_k$. [So, f is a n_1-th power of a polynomial with simple roots.] Moreover, if K' is any other field where f splits, and n is the common exponent above [e.g, $n = n_1$], we must have $f = c(X - \alpha_1')^n \cdots (X - \alpha_k')^n$ in $K'[X]$. [I.e., the number of distinct roots k and the exponent n are the same.]*

Corollary 1.2.12. *If $f \in F[x]$ is irreducible and* $\mathrm{char}(F) = 0$ *[or $f' \neq 0$], then f has only simple roots [in any extension of F].*

Theorem 1.2.13. (1) *If $\phi : F \to K$ is an embedding of F, K is algebraically closed and α is algebraic over F, then the number of extensions of ϕ to $F[\alpha]$ is equal to the number of distinct roots of $\min_{\alpha,F}(X)$.*

(2) *If K/F is an algebraic extension, $\phi : F \to L$, with L algebraically closed, then there exists an extension $\psi : K \to L$ of ϕ. Moreover, if K is also algebraically closed and $L/\phi(F)$ is algebraic, then ψ is an isomorphism. [Hence the algebraic closure of a field is unique up to isomorphism, and we denote the algebraic closure of F by $\bar{F}$.]*

(3) *If K/F is an algebraic extension and $\bar{K}$ is an algebraic closure of K, then it is also an algebraic closure of F. Conversely, if $\bar{F}$ is an algebraic closure of F and K' is the image of the embedding of K into $\bar{F}$, then $\bar{F}$ is an algebraic closure of K'.*

1.3. **Splitting Fields.**

Definition 1.3.1. K is a *splitting field* of $f \in F[X]$ if $f(X)$ splits in K, but not in any proper subfield of K. In particular if f splits in an extension of F as $f = c(X - \alpha_1) \cdots (X - \alpha_n)$, then $F[\alpha_1, \dots, \alpha_n]$ is a splitting field of f.

Theorem 1.3.2. *If K_1/F and K_2/F are two splitting fields of $f \in F[X]$ [or of the same* families *of polynomials] in different algebraic closure [so that they are distinct], then there exists an isomorphism between K_1 and K_2 over F [induced by the isomorphism of the algebraic closures].*

Remark 1.3.3. If $\bar{F}$ is an algebraic closure of F and $\alpha_1, \dots, \alpha_n \in \bar{F}$ are *all* the roots of $f(X)$, then the splitting field of F is $F[\alpha_1, \dots, \alpha_n]$.

Definition 1.3.4. K is *normal extension* of F is it is algebraic over F and any embedding $\phi : K \to \bar{K} = \bar{F}$ over F is an automorphism of K.

Theorem 1.3.5. *Let $F \subseteq K \subseteq \bar{F}$. The following are equivalent:*

1. *K is normal.*
2. *K is a splitting field of a family of polynomials.*
3. *Every polynomials in $F[X]$ that has a root in K, splits in $K[X]$.*

Theorem 1.3.6. *The class of normal extensions is* quasi-*distinguished [but not distinguished]. Also, if K_1/F and K_2/F are normal, then so is $K_1 \cap K_2/F$.*

Proposition 1.3.7. *If $[K : F] = 2$, then K/F is normal.*

Remark 1.3.8.
1. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ are normal extensions, but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is *not* normal.
2. $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$, where $\zeta_3 = \mathrm{e}^{2\pi\mathrm{i}/3}$, is normal, and $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\zeta_3, \sqrt[3]{2})$, but $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is *not* normal.

1.4. Separable Extensions.

Lemma 1.4.1. *Let $\sigma : F \to L$ and $\tau : F \to L'$ be embeddings of F into algebraically closed fields, and let K/F be an algebraic extension. Then, the number [or cardinality] of extensions of σ to K is the same as the number of extensions of τ to K.*

Definition 1.4.2.
1. Let K/F be a finite extension and $\bar{F}$ be an algebraic closure of F. Then, the *separable degree* of K/F is
$$[K : F]_{\mathrm{s}} \stackrel{\text{def}}{=} \text{number of embeddings } \phi : K \to \bar{F} \text{ over } F.$$
2. A polynomial $f \in F[X]$ is a *separable polynomial* if it has no multiple roots.
3. Let α be algebraic over F. Then α is *separable* over F if $\min_{\alpha,F}(X)$ is separable.
4. K/F is a *separable extension* if every element of K is separable over F.

Remark 1.4.3. If $\phi : F \to L$ is embedding of F and L is algebraic closed, then
$$[K : F]_{\mathrm{s}} = \text{number of extensions } \psi : K \to L \text{ of } \phi.$$

Theorem 1.4.4. *If L/K and K/F are algebraic extensions, then*

$$[L : F]_{\mathrm{s}} = [L : K]_{\mathrm{s}} \cdot [K : F]_{\mathrm{s}}.$$

Moreover, if $[L : F] < \infty$, then

$$[L : F]_{\mathrm{s}} \leq [L : F],$$

and K/F is separable if, and only if, $[L : F]_{\mathrm{s}} = [L : F]$.

Theorem 1.4.5. *If $K = F[\{\alpha_i \ : \ i \in I\}]$, where I is a set of indices and α_i is separable over F for all $i \in I$, then K/F is separable.*

Theorem 1.4.6. *The class of separable extensions is distinguished.*

Proposition 1.4.7. *Let K be a finite extension of F inside $\bar{F}$. Then the smallest extension of K which is normal over F is $L \overset{\text{def}}{=} \phi_1(K) \ldots \phi_n(K)$, where $\{\phi_1, \ldots, \phi_n\}$ are* all *the embeddings of K into $\bar{F}$ over F. (The $\phi_i(K)$'s are called the* conjugates *of K.) Moreover, if K/F is separable, then L is also separable over F.*

Definition 1.4.8.
(1) The field L in the proposition above is called the *normal closure* of K/F.
(2) Let

$$F^{\mathrm{s}} \overset{\text{def}}{=} \text{compositum of all separable extensions of } F.$$

F^{s} is called the *separable closure* of all F.
(3) If $K = F[\alpha]$, then K is said to be a *simple extension* of F.

Theorem 1.4.9 (Primitive Element Theorem)**.** *If $[F : F] < \infty$, then K/F has a primitive element if, and only if, there are finitely many* intermediate fields *(i.e., fields L such that $F \subseteq L \subseteq K$). Moreover, if K/F is (finite and) separable, then K/F has a primitive element.*

Lemma 1.4.10. *If $f \in F[X]$ is* irreducible*, then f has distinct roots if, and only if, $f'(X)$ is a non-zero polynomial.*

Proposition 1.4.11. (1) *α is separable over F if, and only if,* $(\min_{\alpha,F})' \not\equiv 0$.
(2) *If* $\mathrm{char}(F) = 0$*, then any extension of F is separable.*
(3) *Let* $\mathrm{char}(F) = p > 0$*. Then α is inseparable over F if, and only if,* $\min_{\alpha,F} \in F[X^p]$*. (And thus,* $\min_{\alpha,F}$ *is a p-power in $\bar{F}[X]$.)*

1.5. **Inseparable Extensions.**

Definition 1.5.1. An algebraic extension K/F is *inseparable* if it is not separable. (Note that if K/F is inseparable, then $\mathrm{char}(F) = p > 0$.)

Proposition 1.5.2. *If $F[\alpha]/F$ is finite and inseparable, then* $\min_{\alpha,F}(X) = f(X^{p^k})$*, where* $p = \mathrm{char}(F)$ *[necessarily positive], for some positive integer k and* separable and irreducible *polynomial $f \in F[X]$. Moreover,* $[F[\alpha] : F]_{\mathrm{s}} = \deg f$*,* $[F[\alpha] : F] = p^k \cdot \deg f$*, and α^{p^k} is separable over F.*

Corollary 1.5.3. *If K/F is finite, then* $[K : F]_{\mathrm{s}} \mid [K : F]$*. If* $\mathrm{char}(F) = 0$*, then the quotient is* 1*, and if* $\mathrm{char}(F) = p > 0$*, then the quotient is a power of p.*

Definition 1.5.4. Let K/F be a finite algebraic extension. The inseparable degree of K/F is

$$[K : F]_{\mathrm{i}} \stackrel{\text{def}}{=} \frac{[K : F]}{[K : F]_{\mathrm{s}}}.$$

Proposition 1.5.5. *Let K/F be a finite algebraic extension. Then:*
(1) *K/F is separable if, and only if,* $[K : F]_{\mathrm{i}} = 1$*;*
(2) *if E is an intermediate field, then* $[K : F]_{\mathrm{i}} = [K : E]_{\mathrm{i}} \cdot [E : F]_{\mathrm{i}}$.

Definition 1.5.6. (1) Let α be algebraic over F, with $\mathrm{char}(F) = p$. We say that α is *purely inseparable* over F if $\alpha^{p^n} \in F$ for some positive integer n. [Thus, $\min_{\alpha,F} \mid X^{p^n} - \alpha^{p^n} = (X - \alpha)^{p^n}$.]
(2) An algebraic [maybe infinite] extension K/F is a *purely inseparable extension* if $[K : F]_{\mathrm{s}} = 1$.

Proposition 1.5.7. *An element α is purely inseparable if, and only if,* $\min_{\alpha,F}(X) = X^{p^n} - a$ *for some positive integer n and $a \in F$. [Observe that $a = \alpha^{p^n}$.]*

Proposition 1.5.8. *Let K/F be an algebraic extension. The following are equivalent:*

1. *K/F is purely inseparable [i.e., $[K : F]_{\mathrm{s}} = 1$].*
2. *All elements of K are purely inseparable over F.*
3. *$K = F[\alpha_i \; : \; i \in I]$, for some set of indices I, with α_i purely inseparable over F.*

Proposition 1.5.9. *The class of purely inseparable extensions is distinguished.*

Definition 1.5.10. (1) Let F be a field and G be a subgroup of $\mathrm{Aut}(F)$. Then:

$$F^G \stackrel{\text{def}}{=} \{\alpha \in F \; : \; \phi(\alpha) = \alpha, \, \forall \phi \in G\},$$

is the *fixed field* of G. (**Note:** it is a field.)

(2) The extension K/F is a *Galois extension* if it is normal and separable. In this case, the *Galois group* of K/F, denoted by $\mathrm{Gal}(K/F)$ is the group of automorphisms of K over F [i.e., automorphisms of K which fix F].

Remark 1.5.11. If K/F is Galois, then $\mathrm{Gal}(K/F)$ is equal to the set of embeddings of K into $\bar{K}$. Also, if K/F is finite, then K/F is Galois if, and only if, $|\mathrm{Aut}_F(K)| = [K : F]$, and so $|\mathrm{Gal}(K/F)| = [K : F]$.

Remark 1.5.12. Note that for any field extension K/F we have a group of automorphisms over F, which we denote by $\mathrm{Aut}_F(K)$. But, usually, the notation $\mathrm{Gal}(K/F)$ is reserved for Galois extensions only. [A few authors do use $\mathrm{Gal}(K/F)$ for $\mathrm{Aut}_F(K)$, though.]

Proposition 1.5.13. *Let K/F be an algebraic extension. Then*

$$K' \stackrel{\text{def}}{=} \{x \in K \; : \; x \textit{ is separable over } F\}$$

is a field [equal to the compositum of all separable extensions of F that are contained in K]. [So, it is clearly the maximal *separable extension of F contained in K.] Then, K'/F is separable and K/K' is purely inseparable.*

Corollary 1.5.14. (1) *K/F is separable and purely inseparable, then $K = F$.*

(2) *If α is separable and purely inseparable over F, then $\alpha \in F$.*

Corollary 1.5.15. *If K/F is normal, then the maximal separable extension of F contained in K [i.e., the K' in the proposition above] is normal over F. [Hence, K'/F is Galois.]*

Corollary 1.5.16. *If F/E and K/E are finite, with $F, K \subseteq \mathcal{F}$, with F/E separable and K/E purely inseparable, then*

$$[F\,K : K] = [F : E] = [F\,K : E]_{\mathrm{s}},$$
$$[F\,K : F] = [K : E] = [F\,K : E]_{\mathrm{i}}.$$

Definition 1.5.17. Let F be a field [or a ring] of characteristic p, with p prime. The *Frobenius morphism* of F is the map

$$\begin{aligned} \sigma : F &\to F \\ x &\mapsto x^p. \end{aligned}$$

Corollary 1.5.18. *Let K/F be a* finite *extension in characteristic $p > 0$ and σ be the Frobenius.*

(1) *If $K^\sigma F = K$, then K/F is separable, where*

$$K^\sigma = \sigma(K) = \{\sigma(x) \;:\; x \in K\}.$$

(2) *If K/F is separable, then $K^{\sigma^n} F = K$ for any positive integer n.*

Remark 1.5.19. (1) If $K = F[\alpha_1, \ldots, \alpha_m]$, then $K^{\sigma^n} F = F[\alpha_1^{p^n}, \ldots, \alpha_m^{p^n}]$.

(2) Notice that if K/F is an algebraic extension, we can always have an intermediate field K' such that K'/F is separable and K/K' is purely inseparable, but not always we can have a K'' such that K''/F is purely inseparable and K/K'' is separable. [For example, take $F = \mathbb{F}_p(s, t)$, with $p > 2$, and $K = F[\alpha]$, where α is a root of $X^p - \beta$ and β is a root of $X^2 - s\,X + t$.]

The next proposition states that if K/F is normal, then there is such a K''.

Proposition 1.5.20. *Let K/F be* normal *and $G \stackrel{\text{def}}{=} \mathrm{Aut}_F(K)$ [where $\mathrm{Aut}_F(K)$ is the set of automorphisms of K over F] and K^G be the fixed field of G [as in Definition 1.5.10]. Then K^G/F is purely inseparable and K/K^G is separable. [Hence, K/K^G is Galois.]*

Moreover, if K' is the maximal separable extension of F contained in K, then $K = K' K^G$ and $K' \cap K^G = F$.

Definition 1.5.21. A field F is a *perfect field* if either $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) = p > 0$ and the Frobenius $\sigma : F \to F$ is onto [or equivalently, every element of F has a p-th root]. [Note that σ is always injective, so σ is, in in this case, an automorphism of F.]

Proposition 1.5.22. *Every algebraic extension of a perfect field F is both perfect and separable over F.*

1.6. **Finite Fields.**

Theorem 1.6.1. *If F is a field with q [finite] elements, then:*

(1) $\mathrm{char}(F) = p > 0$ *and so* $\mathbb{F}_p \subseteq F$*;*
(2) $q = p^n$ *for some positive integer n;*
(3) *F is the splitting field of $X^q - X$ (over $\mathbb{F}_p$);*
(4) *any other field with q elements is isomorphic to F, and in a fixed algebraic closure of $\mathbb{F}_p$, there exists only one field with q elements, usually denoted by $\mathbb{F}_q$;*
(5) *there exists $\xi \in F$, such that $F^\times = \langle \xi \rangle$;*
(6) *for any positive integer r, there is a unique field with p^r elements in a fixed algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, which is the unique extension of $\mathbb{F}_p$ of degree r in $\bar{\mathbb{F}}_p$.*

Proposition 1.6.2. *Any algebraic extension of a finite field Galois [i.e., it is both normal and separable].*

Proposition 1.6.3. *The set of automorphisms of $\mathbb{F}_{p^r}$ is $\{\mathrm{id}, \sigma, \sigma^2, \dots, \sigma^{r-1}\}$, where σ is the Frobenius map. [Note that these are* all *automorphisms, and they are automorphisms over $\mathbb{F}_p$.]*

Proposition 1.6.4. *$\mathbb{F}_{p^s}$ is an extension of $\mathbb{F}_{p^r}$ if, and only if, $r \mid s$. In this case, the set of embeddings of $\mathbb{F}_{p^s}$ into $\bar{\mathbb{F}}_p$ over $\mathbb{F}_{p^r}$ [or equivalently, since normal, the set of automorphisms of $\mathbb{F}_{p^s}$ over $\mathbb{F}_{p^r}$] is $\{\mathrm{id}, \sigma^r, \sigma^{2r}, \dots, \sigma^{s-r}\}$, where σ is the Frobenius map. [In other words, $\mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_{p^r}) = \langle \sigma^r \rangle$.]*

Proposition 1.6.5. *The algebraic closure $\bar{\mathbb{F}}_p$ is $\displaystyle\bigcup_{r>0} \mathbb{F}_{p^r}$. [Note that any finite union is contained in a single finite field.]*

2. Galois Theory

2.1. Galois Extensions.

Proposition 2.1.1. *Galois extensions form a quasi-distinguished class, and if K_1/F and K_2/F are Galois, then so is $K_1 \cap K_2/F$.*

Theorem 2.1.2. *Let K/F be a Galois extension and $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$. Then*

1. *$K^G = F$;*
2. *if E is an intermediate field ($F \subseteq E \subseteq K$), then K/E is also Galois;*
3. *the map $E \mapsto \mathrm{Gal}(K/E)$ is injective.*

Corollary 2.1.3. *Let K/F be a Galois extension and $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$. If E_i is an intermediate field and $H_i \stackrel{\text{def}}{=} \mathrm{Gal}(K/E_i)$, for $i = 1, 2$, then:*

1. *$H_1 \cap H_2 = \mathrm{Gal}(K/E_1 E_2)$;*
2. *if $H = \langle H_1, H_2 \rangle$ [i.e., H is the smallest subgroup of G containing H_1 and H_2], then $K^H = E_1 \cap E_2$.*

Corollary 2.1.4. *Let K/F be separable and* ***finite****, and L be the normal closure of K/F [i.e., the smallest normal extension of F containing K]. Then L/F is finite and Galois.*

Lemma 2.1.5. *Let K/F be a separable extension such that for all $\alpha \in K$, $[F[\alpha] : F] \leq n$, for some fixed n. Then $[K : F] \leq n$.*

Theorem 2.1.6 (Artin)**.** *Let K be a field, G be a subgroup of* $\mathrm{Aut}(K)$ *with $|G| = n < \infty$, and $F \stackrel{\text{def}}{=} K^G$. Then K/F is Galois and $G = \mathrm{Gal}(K/F)$ (and $[K : F] = n$).*

Corollary 2.1.7. *Let K/F be Galois and **finite** and $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$. Then, for any subgroup H of G, $H = \mathrm{Gal}(K/K^H)$.*

Remark 2.1.8. The above corollary is not true if the extension is infinite! The map $H \mapsto K^H$ is not injective! For example, $\bar{\mathbb{F}}_p/\mathbb{F}_p$ is Galois, the cyclic group H generated by the Frobenius is not the Galois group, and yet $K^H = \mathbb{F}_p$.

Lemma 2.1.9. *Let K_1 and K_2 be two extensions of F with $\phi : K_1 \to K_2$ an* isomorphism over F. *Then $\mathrm{Aut}_F(K_2) = \phi \circ \mathrm{Aut}_F(K_1) \circ \phi^{-1}$.*

Theorem 2.1.10. *Let K/F be a Galois extension and $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$. If E is an intermediate extension, then E/F is normal [and thus Galois] if, and only if, $H \stackrel{\text{def}}{=} \mathrm{Gal}(K/E)$ is a normal subgroup of G. In this case, $\phi \mapsto \phi|_E$ induces an isomorphism between G/H and $\mathrm{Gal}(E/F)$.*

Definition 2.1.11. An extension K/F is an *Abelian extension* (resp., a *cyclic extension*) if it is Galois and $\mathrm{Gal}(K/F)$ is Abelian (resp., cyclic).

Corollary 2.1.12. *If K/F is Abelian (resp., cyclic), then for any intermediate field E, K/E and E/F are Abelian (resp., cyclic).*

Theorem 2.1.13 (Fundamental Theorem of Galois Theory)**.** *Let K/F be **finite** and Galois, with $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$. The results above gives: the map*

$$\begin{aligned} \{\textit{subgroups of } G\} &\longrightarrow \{\textit{intermediate fields of } K/F\} \\ H &\longmapsto K^H \end{aligned}$$

is a bijection with inverse

$$\begin{array}{rcl} \{\textit{intermediate fields of } K/F\} & \longrightarrow & \{\textit{subgroups of } G\} \\ E & \longmapsto & \mathrm{Gal}(K/E). \end{array}$$

Moreover an intermediate field E is Galois if, and only if, $H \stackrel{\text{def}}{=} \mathrm{Gal}(K/E)$ is normal in G, and $\mathrm{Gal}(E/F) \cong G/H$, induced by $\phi \mapsto \phi|_E$.

Remark 2.1.14. Note that the maps $H \mapsto K^H$ and $E \mapsto \mathrm{Gal}(K/E)$ are *inclusion reversing*, i.e., $H_1 \leq H_2$ implies $K^{H_1} \supseteq K^{H_2}$, and if $E_1 \subseteq E_2$, then $\mathrm{Gal}(K/E_1) \geq \mathrm{Gal}(K/E_2)$.

Theorem 2.1.15 (Natural Irrationalities)**.** *Let K/F be a Galois extension and L/F be an arbitrary extension, with $K, L \subseteq \mathcal{F}$ [so that we can consider the compositum $L\,K$]. Then $K\,L$ is Galois over L and K is Galois over $K \cap L$. Moreover, if $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$ and $H \stackrel{\text{def}}{=} \mathrm{Gal}(K\,L/L)$, then for any $\phi \in H$, $\phi|_K \in G$ and $\phi \mapsto \phi|_K$ is an isomorphism between H and $\mathrm{Gal}(K/K \cap L)$.*

Corollary 2.1.16. *If K/F is finite and Galois and L/F is an arbitrary extension, then $[K\,L : L] \mid [K : F]$.*

Remark 2.1.17. The above theorem does not hold for if K/F is not Galois. For example, $F \stackrel{\text{def}}{=} \mathbb{Q}$, $K \stackrel{\text{def}}{=} \mathbb{Q}(\sqrt[3]{2})$ and $L \stackrel{\text{def}}{=} \mathbb{Q}(\zeta_3 \sqrt[3]{2})$, where $\zeta_3 = \mathrm{e}^{2\pi \mathrm{i}/3}$.

Theorem 2.1.18. *Let K_1/F and K_2/F be Galois extensions with $K_1, K_2 \in \mathcal{F}$. Then $K_1\,K_2/F$ is Galois. Moreover, if $G \stackrel{\text{def}}{=} \mathrm{Gal}(K_1\,K_2/F)$, $G_1 \stackrel{\text{def}}{=} \mathrm{Gal}(K_1/F)$, $G_2 \stackrel{\text{def}}{=} \mathrm{Gal}(K_2/F)$ and*

$$\begin{array}{rcl} \Phi : G & \to & G_1 \times G_2 \\ \phi & \mapsto & (\phi|_{K_1}, \phi|_{K_2}), \end{array}$$

then Φ is injective and if $K_1 \cap K_2 = F$, then Φ is an isomorphism.

Corollary 2.1.19. *If K_i/F is Galois and $G_i \stackrel{\text{def}}{=} \mathrm{Gal}(K_i/F)$ for $i = 1, \ldots, n$ and $K_{i+1} \cap (K_1 \ldots K_i) = F$ for $i = 1, \ldots, (n-1)$, then $\mathrm{Gal}(K_1 \ldots K_n/F) = G_1 \times \cdots \times G_n$.*

Corollary 2.1.20. *Let K/F be finite and Galois, with $G \stackrel{\text{def}}{=} \operatorname{Gal}(K/F) = G_1 \times \cdots \times G_n$, $H_i \stackrel{\text{def}}{=} G_1 \times \cdots \times G_{i-1} \times 1 \times G_{i+1} \times \cdots \times G_n$ and $K_i \stackrel{\text{def}}{=} K^{H_i}$. Then K_i/F is Galois with* $\operatorname{Gal}(K_i/F) \cong G_i$*, $K_{i+1} \cap (K_1 \dots K_i) = F$ and $K = K_1 \dots K_n$.*

Corollary 2.1.21. *Abelian extensions are* quasi-*distinguished [see Definition 1.1.22]. Moreover, if K is an Abelian extension of F and E is an intermediate field, then E/F is also Abelian. [Hence, intersections of Abelian extensions are also Abelian.]*

Remark 2.1.22. Observe that, as with Galois extensions [and Abelian extensions are Galois by definition], we do *not* always have that if K/E and E/F are Abelian, then K/F is Abelian. For example, $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are Abelian (since they are degree two extensions), but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not even Galois [since $X^4 - 2$ does not split in $\mathbb{Q}(\sqrt[4]{2})$].

2.2. **Examples and Applications.**

Definition 2.2.1. The *Galois group of a separable polynomial* $f \in F[X]$ is the Galois group of the splitting field of f over F. We will denote it by G_f or $G_{f,F}$.

Proposition 2.2.2.
(1) *Let $f \in F[X]$ be a [not necessarily separable or irreducible] polynomial, K be its splitting field, and n be the number of* distinct *roots of f [in K]. Then, $G \stackrel{\text{def}}{=} \operatorname{Aut}_F(K)$ is a subgroup of the symmetric group S_n, seen as permutations of the roots of f. [In particular, any $\sigma \in G$ is determined by its values on the roots of f, and hence, if $\sigma \in G$ fixes all roots of f, then $\sigma = \mathrm{id}_K$.]*

(2) *If $f \in F[X]$ is* irreducible *[but not necessarily separable] and K, n, and G are as above, then G is a* transitive subgroup *of S_n [i.e., for all $i, j \in \{1, \dots, n\}$, there is $\sigma \in G$ such that $\sigma(i) = j$.]*

(3) *Let K/F be Galois [and hence* separable*] with $G \stackrel{\text{def}}{=} \operatorname{Gal}(K/F)$, $\alpha \in K$,*
$$\mathcal{O} \stackrel{\text{def}}{=} \{\sigma(\alpha) \ : \ \sigma \in G\}$$
be the orbit *of α by the action of G in K. Then, $\mathcal{O}$ is finite, say, $\mathcal{O} = \{\alpha_1, \dots, \alpha_k\}$, and*
$$\min_{\alpha,F} = (x - \alpha_1) \cdots (x - \alpha_k).$$

Note that $|\mathcal{O}| \mid [K : F] = |G|$.

(4) *Let* K/F *be* finite *and Galois with* $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/F)$, *and let* $\alpha \in K$. *Then,* $K = F[\alpha]$ *if, and only if, the orbit of* α *by* G *has exactly* $[K : F]$ *elements.*

Proposition 2.2.3 (Quadratic Extensions)**.**

(1) *If* $\mathrm{char}(F) \neq 2$ *and* $[K : F] = 2$, *then there exists an* $a \in F$ *such that* $K = F[\alpha]$, *with* $\min_{\alpha,F} = X^2 - a$. *Also,* $\mathrm{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z}$ *and the non-identity element is such that* $\phi(\alpha) = -\alpha$.

(2) *If* $f \in F[X]$ *is a quadratic separable polynomial, then the splitting field of* F *has degree two over* F, $G_f \cong \mathbb{Z}/2\mathbb{Z}$ *and the non-zero element of* G_f *is takes a root of* f *to the other root.*

Definition 2.2.4. Let $f \in F[X]$, such that

$$f(X) = \prod_{i=1}^{n} (X - \alpha_i).$$

Then the *discriminant* of f is defined as

$$\Delta_f = \Delta \stackrel{\text{def}}{=} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

Proposition 2.2.5. *For any* $f \in F[X]$, $\Delta_f \in F$. *In particular if* $f = aX^2 + bX + c$, *then* $\Delta_f = b^2 - 4ac$ *and if* $f = X^3 + aX + b$, *then* $\Delta_f = -4a^3 - 27b^2$.

Proposition 2.2.6 (Cubic Extensions and Polynomials)**.**

(1) *If* $[K : F] = 3$, *then for any* $\alpha \in K - F$, *we have* $K = F[\alpha]$.

(2) *If* $\mathrm{char}(F) \neq 3$ *and* $f \in F[X]$ *is irreducible of degree* 3, *say* $f(X) = X^3 + aX^2 + bX + c$, *then the splitting field of* f *is the same as the splitting field of the polynomial* $\tilde{f}(X) \stackrel{\text{def}}{=} f(X - a/3) = X^3 + \tilde{a}X + \tilde{b}$. *[Hence* $G_f = G_{\tilde{f}}$.*]*

(3) *If the splitting field of a separable* $f \in F[X]$ *is of degree* 3, *then* $G_f \cong \mathbb{Z}/3\mathbb{Z}$ *and if* $\alpha_1, \alpha_2, \alpha_3$ *are the [distinct] roots of* f, *then* $G_f = \langle \phi \rangle$, *where* $\phi(\alpha_1) = \alpha_2$ *and* $\phi(\alpha_2) = \alpha_3$ *and* $\phi(\alpha_3) = \alpha_1$. *Note that in this case,* $G_f \cong \mathrm{A}_3$, *where* A_n *is the* alternating subgroup *of* S_n *[i.e., the subgroup of* even *permutations].*

(4) *If the splitting field of a separable* $f \in F[X]$ *is* not *of degree* 3*, then* $G_f \cong \mathrm{S}_3$ *[and hence* G_f *can permute the roots of* f *in all possible ways].*

(5) *Let* $f = \prod_{i=1}^{3}(X - \alpha_i) \in F[X]$ *and*

$$\delta \stackrel{\text{def}}{=} (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3).$$

[Thus, $\delta^2 = \Delta_f$*.] If* f *is irreducible in* $F[X]$*,* $\Delta_f \neq 0$ *[i.e.,* f *is separable] and* $\mathrm{char}(F) \neq 2$*, then* $G_f \cong \mathrm{S}_3$ *if, and only if,* $\delta \notin F$ *[or equivalently,* Δ_f *is not a square in* F*.] [Note that if* $\delta \notin F$*, then* $F[\delta]/F$ *is a degree two extension contained in the splitting field of* f*.]*

Examples 2.2.7. From the above, we can deduce:

(1) If $f \stackrel{\text{def}}{=} X^3 - X + 1 \in \mathbb{Q}[X]$, then $\Delta_f = -23$, and hence $G_f = \mathrm{S}_3$.

(2) If $f \stackrel{\text{def}}{=} X^3 - 3X + 1 \in \mathbb{Q}[X]$, then $\Delta_f = 81$, and hence $G_f = \mathbb{Z}/3\mathbb{Z}$.

Example 2.2.8. If $f = X^4 - 2 \in \mathbb{Q}[X]$, then $G_f \cong \mathrm{D}_8$, the dihedral group of 8 elements. More precisely, if $\phi \in \mathrm{Gal}(Q[\sqrt[4]{2}, \mathrm{i}]/\mathbb{Q}[\mathrm{i}])$ such that $\phi(\sqrt[4]{2}) = \sqrt[4]{2}\,\mathrm{i}$ and $\psi \in \mathrm{Gal}(Q[\sqrt[4]{2}, \mathrm{i}]/\mathbb{Q}[\sqrt[4]{2}])$ such that $\psi(\mathrm{i}) = -\mathrm{i}$ [i.e., ψ is the complex conjugation], then

$$\begin{aligned} G_f &= \left\langle \phi, \psi \ : \ \phi^4 = \mathrm{id},\ \psi^2 = \mathrm{id},\ \psi \circ \phi = \phi^3 \circ \psi \right\rangle \\ &= \left\{ \mathrm{id},\ \phi,\ \phi^2,\ \phi^3,\ \psi,\ \phi \circ \psi,\ \phi^2 \circ \psi,\ \phi^3 \circ \psi \right\}. \end{aligned}$$

Proposition 2.2.9. *Let* E *be a field,* $t_1, \ldots, t_n$ *be algebraically independent variables over* E*,* $s_1, \ldots, s_n$ *be their elementary symmetric functions,* $F \stackrel{\text{def}}{=} E(s_1, \ldots, s_n)$ *and* $K \stackrel{\text{def}}{=} E(t_1, \ldots, t_n)$*. Then* $\min_{t_i, F} = \prod_{i=1}^{n}(X - t_i)$ *and* $\mathrm{Gal}(K/F) \cong \mathrm{S}_n$*.*

Theorem 2.2.10 (Fundamental Theorem of Algebra)**.** $\mathbb{C}$ *is the algebraic closure of* $\mathbb{R}$*.*

Lemma 2.2.11. *If* $G \subseteq \mathrm{S}_p$*, with* p *prime, and* G *contains a transposition and a* p*-cycle, then* $G = \mathrm{S}_p$*.*

Proposition 2.2.12. *If $f \in \mathbb{Q}[X]$ is irreducible, $\deg f = p$, with p prime, and if f has exactly two complex roots, then $G_f \cong \mathrm{S}_p$.*

Example 2.2.13. As an application of the proposition above, let $f \stackrel{\text{def}}{=} X^5 - 4X + 2 \in \mathbb{Q}[X]$. Then $G_f \cong \mathrm{S}_5$. In fact, one can use the above proposition to prove that for every prime p there is a polynomial $f_p \in \mathbb{Q}[X]$ such that $G_{f_p,\mathbb{Q}} = \mathrm{S}_p$. [One can get all S_n, in fact, but it is harder.]

Theorem 2.2.14. *Let $f \in \mathbb{Z}[X]$ be a monic separable polynomial, p be a prime that does not divide the discriminant of f, and $\bar{f} \in \mathbb{Z}/p\mathbb{Z}[X]$ be the reduction modulo p of f [i.e., obtained by reducing the coefficients]. Then, there is a bijection between the roots of f and the roots of $\bar{f}$, denoted by $\alpha \mapsto \bar{\alpha}$, and an injection $i : G_{\bar{f}} \to G_f$, such that, if $\phi \in G_{\bar{f}}$ and $\bar{\alpha}_i$ and $\bar{\alpha}_j$ are roots of $\bar{f}$, with $\phi(\bar{\alpha}_i) = \bar{\alpha}_j$, then $i(\phi)(\alpha_i) = \alpha_j$.*

In particular, if $\phi \in G_{\bar{f}}$, then G_f has an element [namely $i(\phi)$] that has the same cycle structure [seen as a permutation] as ϕ itself. [E.g., if ϕ as a permutation is a product of a two-cycle, a 4-cycle and a 7-cycle [all disjoint], then $i(\phi)$ is also a product of a two-cycle, a 4-cycle and a 7-cycle [all disjoint] in G_f.]

Example 2.2.15. As an application of the theorem above, one can prove that $f \stackrel{\text{def}}{=} X^5 - X - 1 \in \mathbb{Z}[X]$ is such that $G_f = \mathrm{S}_5$, by reducing f modulo 5 and modulo 2.

2.3. **Roots of Unity.**

Definition 2.3.1.

(1) A n-th *root of unity* in a field F is a root of $X^n - 1$ in F. A *root of unity* [with no n specified] is a root of unit for some n.
(2) The set of all roots of unity form an Abelian group, denoted by $\boldsymbol{\mu}(F)$ or simply $\boldsymbol{\mu}$.
(3) The set of n-th roots of unity in F is a *cyclic* group denoted by $\boldsymbol{\mu}_n(F)$ or simply $\boldsymbol{\mu}_n$.
(4) If $\mathrm{char}(F) \nmid n$, then $|\boldsymbol{\mu}_n| = n$ and a generator of $\boldsymbol{\mu}_n$ is called a *primitive n-th root of unity*.

Proposition 2.3.2. (1) *If* $\operatorname{char}(F) = p > 0$, $n = p^r m$, *and* $p \nmid m$, *then* $\boldsymbol{\mu}_n(F) = \boldsymbol{\mu}_m(F)$ *[and so* $|\boldsymbol{\mu}_n(F)| = m$*].*

(2) *If* $\gcd(n, m) = 1$, *then* $\boldsymbol{\mu}_n \times \boldsymbol{\mu}_m \cong \boldsymbol{\mu}_n \cdot \boldsymbol{\mu}_m = \boldsymbol{\mu}_{nm}$ *and the isomorphism is given by* $(\zeta, \zeta') \mapsto \zeta\,\zeta'$. *[In particular, if* ζ_n *and* ζ_m *are primitive* n*-th and* m*-th roots of unity, then* $\zeta_n\zeta_m$ *is a primitive* nm*-th root of unity.]*

Proposition 2.3.3. *Let* F *be a field such that* $\operatorname{char}(F) \nmid n$, *and* ζ_n *a primitive* n*-th root of unity. Then* $F[\zeta_n]/F$ *is Galois. If* $\phi \in \operatorname{Gal}(F[\zeta_n]/F)$, *then* $\phi(\zeta_n) = \zeta_n^{i(\phi)}$, *for some* $i(\phi) \in (\mathbb{Z}/n\mathbb{Z})^\times$ *and this map* $i : \operatorname{Gal}(F[\zeta_n]/F) \to (\mathbb{Z}/n\mathbb{Z})^\times$ *is injective. Thus,* $\operatorname{Gal}(F[\zeta_n]/F)$ *is Abelian.*

Remark 2.3.4. Note that $\operatorname{Gal}(F[\zeta_n]/F)$ is not necessarily cyclic. For example, $\operatorname{Gal}(\mathbb{Q}[\zeta_8]/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Definition 2.3.5. We say that K/F is a *cyclotomic extension* if there exists a root of unity ζ over F such that $K = F[\zeta]$. [*Careful*: in Lang, an extension is cyclotomic if there exists a root of unity ζ over F such that $K \subseteq F[\zeta]$!]

Definition 2.3.6. Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ denote the *Euler phi-function*, which is defined as

$$\varphi(n) \stackrel{\text{def}}{=} |\{m \in \mathbb{Z} \;:\; 0 < m < n \text{ and } \gcd(m, n) = 1\}|\,.$$

Theorem 2.3.7. *If* ζ_n *is a primitive* n*-th root of unity in* $\mathbb{Q}$, *then* $[\mathbb{Q}[\zeta_n] : \mathbb{Q}] = \varphi(n)$ *and the map* $i : \operatorname{Gal}(F[\zeta_n]/F) \to (\mathbb{Z}/n\mathbb{Z})^\times$ *[as in Proposition 2.3.3] is an isomorphism.*

Corollary 2.3.8. *If* ζ_m *and* ζ_n *are a primitive* m*-th root of unity and primitive* n*-th root of unity, respectively, with* $\gcd(m, n) = 1$,*then* $\mathbb{Q}[\zeta_m] \cap \mathbb{Q}[\zeta_n] = \mathbb{Q}$,

Remark 2.3.9. If $m = \operatorname{lcm}(n_1, \dots, n_r)$, and ζ_{n_i} is a primitive n_i-th root of unity for $i = 1, \dots, r$, then $\mathbb{Q}[\zeta_{n_1}] \cdots \mathbb{Q}[\zeta_{n_r}] = \mathbb{Q}[\zeta_m]$.

Definition 2.3.10. Let n be a positive integer not divisible by $\operatorname{char}(F)$. The polynomial

$$\Phi_n(X) \stackrel{\text{def}}{=} \prod_{\substack{\zeta \text{ prim. } n\text{-th} \\ \text{root of 1 in } F}} (X - \zeta)$$

is called the n-th *cyclotomic polynomial* [over F].

Proposition 2.3.11.

(1) $\deg \Phi_n = \varphi(n)$.
(2) *If ζ_n is a primitive n-th root of unity, then* $\Phi_n(X) = \min_{\zeta_n,\mathbb{Q}}(X)$.
(3) *If ζ_n is a primitive n-th root of unity, then*

$$\Phi_n(X) = \prod_{\phi \in \operatorname{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q})} (X - \phi(\zeta_n))$$

(4) $X^n - 1 = \displaystyle\prod_{d|n} \Phi_d(X)$.
(5) *If* $\operatorname{char}(F) = 0$, *then* $\Phi_n \in \mathbb{Z}[X]$ *for all* n. *If* $\operatorname{char}(F) = p > 0$, *then* $\Phi_n \in \mathbb{F}_p[X]$ *for all n [not divisible by p].*

Proposition 2.3.12.

(1) *If p is prime, then* $\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$.
(2) *If p is prime, then* $\Phi_{p^r}(X) = \Phi_p(X^{p^{r-1}})$.
(3) *If* $n = p_1^{r_1} \cdots p_s^{r_s}$, *with p_i's distinct primes, then* $\Phi_n(X) = \Phi_{p_1 \cdots p_s}(X^{p_1^{r_1-1} \cdots p_s^{r_s-1}})$.
(4) *If $n > 1$ is odd, then* $\Phi_{2n}(X) = \Phi_n(-X)$.
(5) *If $p \nmid n$, with p an odd prime, then* $\Phi_{pn}(X) = \dfrac{\Phi_n(X^p)}{\Phi_n(X)}$.
(6) *If $p \mid n$, with p prime, then* $\Phi_{pn}(X) = \Phi_n(X^p)$.

Remark 2.3.13. It is *not* true that for all n, the coefficients of $\Phi_n(X)$ are either 0, 1 or -1. The first n for which this fails is $105 = 3 \cdot 5 \cdot 7$.

Theorem 2.3.14 (Dirichlet's Theorem of Primes in Arithmetic Progression)**.** *If* $\gcd(a, r) = 1$, *there are infinitely many primes in the arithmetic progression*

$$a,\ a + r,\ a + 2r,\ a + 3r, \ldots.$$

Theorem 2.3.15. *Given a finite Abelian group G, there exists an extension $F/\mathbb{Q}$ such that* $\mathrm{Gal}(F/\mathbb{Q}) = G$.

Theorem 2.3.16 (Kronecker-Weber)**.** *If $F/\mathbb{Q}$ is finite and Abelian, then there exists a cyclotomic extension $\mathbb{Q}[\zeta]/\mathbb{Q}$ such that $F \subseteq \mathbb{Q}[\zeta]$.*

2.4. **Linear Independence of Characters.**

Definition 2.4.1. Let G be a *monoid* [i.e., a "group" which might not have inverses] and F be a field. A *character* of G in F is a homomorphism $\chi : G \to F^\times$. The *trivial character* is the map constant equal to 1.

Let $f_i : G \to F$ for $i = 1, \ldots, n$. We say that the f_i's are *linearly independent* if

$$\alpha_1 f_1 + \ldots \alpha_n f_n = 0, \quad \alpha_i \in F,$$

then $\alpha_i = 0$ for all i.

Remarks 2.4.2.
1. If K/F is a field extension and $\{\phi_1, \ldots, \phi_n\}$ are the embedding of K over F, then we can think of $\phi|_{K^\times}$ as characters of $K^\times$ in K.
2. If one says only a character in G (without mention of the field), one usually means a character from G in $\mathbb{C}^\times$ or even in

$$S^1 \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} \ : \ |\alpha| = 1\}.$$

Theorem 2.4.3 (Artin)**.** *If $\chi_1, \ldots, \chi_n$* distinct *characters of G in F, then they are linearly independent.*

Corollary 2.4.4. *Let $\alpha_1, \ldots, \alpha_n$ be* distinct *elements of a field $F^\times$. If $a_1, \ldots, a_n \in F$ such that for all positive integer r we have*

$$a_1 \alpha_1^r + \cdots + a_n \alpha_n^r = 0,$$

then $a_i = 0$ for all i.

Corollary 2.4.5. *For any extension K/F, the set $\mathrm{Emb}_{K/F}$ is linearly independent over K.*

2.5. **Norm and Trace.**

Definition 2.5.1. Let K/F be a finite extension, with $[K:F]_{\mathrm{s}} = r$ and $[K:F]_{\mathrm{i}} = p^{\mu}$. [So, $\mathrm{char}(F) = p$ or $[K:F]_{\mathrm{i}} = 1$.] Let $\mathrm{Emb}_{K/F} = \{\phi_1, \ldots, \phi_n\}$ and $\alpha \in K$:

(1) The *norm* of α from K to F is
$$\mathrm{N}_{K/F}(\alpha) \stackrel{\text{def}}{=} \prod_{i=1}^{n} \phi(\alpha^{p^{\mu}}) = \left(\prod_{i=1}^{n} \phi_i(\alpha) \right)^{[K:F]_{\mathrm{i}}}.$$

(2) The *trace* of α from K to F is
$$\mathrm{Tr}_{K/F}(\alpha) \stackrel{\text{def}}{=} [K:F]_{\mathrm{i}} \cdot \sum_{i=1}^{n} \phi_i(\alpha).$$

Remark 2.5.2. Note that if K/F is inseparable, then $\mathrm{Tr}_{K/F}(\alpha) = 0$.

Lemma 2.5.3.

(1) *Let K/F be a finite extension, and $\mathrm{Emb}_{K/F} = \{\phi_1, \ldots, \phi_n\}$ be the set of embeddings of K over F. If L/K is an algebraic extension and $\psi : L \to \bar{F}$ is an embedding over F, then*
$$\{\psi \circ \phi_1, \ldots, \psi \circ \phi_n\} = \mathrm{Emb}_{K/F}.$$

(2) *Let $F \subseteq K \subseteq L$ be field extensions. Let*
$$\mathrm{Emb}_{K/F} = \{\phi_1, \ldots \phi_r\},$$
and
$$\mathrm{Emb}_{L/K} = \{\psi_1, \ldots \psi_s\}.$$
If $\tilde{\phi}_i : \bar{F} \to \bar{F}$ is an extension of ϕ_i to $\bar{F}$ (which exists since $\bar{F}/F$ is algebraic), then
$$\mathrm{Emb}_{L/F} = \{\tilde{\phi}_i \circ \psi_j \ : \ i \in \{1, \ldots, r\} \text{ and } j \in \{1, \ldots, s\}\}.$$

(3) *Let K/F be a* separable *extension. If $\alpha \in K$ is such that $\phi(\alpha) = \alpha$ for all embeddings $\phi \in \mathrm{Emb}_{K/F}$, then $\alpha \in F$.*

Theorem 2.5.4. *Let L/F be a finite extension.*

(1) *For all $\alpha \in K$, $\mathrm{N}_{K/F}(\alpha), \mathrm{Tr}_{K/F}(\alpha) \in F$.*

(2) *If* $[K:F]=n$ *and* $\alpha \in F$, $\mathrm{N}_{K/F}(\alpha)=\alpha^n$ *and* $\mathrm{Tr}_{K/F}(\alpha)=n\cdot\alpha$.

(3) $\mathrm{N}_{K/F}\big|_{K^\times} : K^\times \to F^\times$ *is a [multiplicative] group homomorphism and* $\mathrm{Tr}_{K/F} : K \to F$ *is an [additive] group homomorphism.*

(4) *If* K *is an intermediate field, then*

$$\mathrm{N}_{L/F}=\mathrm{N}_{K/F}\circ \mathrm{N}_{L/K},$$

$$\mathrm{Tr}_{L/F}=\mathrm{N}_{K/F}\circ \mathrm{Tr}_{L/K}.$$

(5) *If* $L=F(\alpha)$, *where* $\min_{,\alpha,F}(X)=X^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0$, *then*

$$\mathrm{N}_{L/F}(\alpha)=(-1)^n a_0, \qquad \mathrm{Tr}_{L/F}(\alpha)=-a_{n-1}.$$

Corollary 2.5.5. *If* $F\subseteq F(\alpha)\subseteq K$, *with* $[K:F]=n$, $\min_{,\alpha,F}(X)=X^d+a_{d-1}X^{d-1}+\cdots+a_1X+a_0$, *and* $[L:F(\alpha)]=e$, *then*

$$\mathrm{N}_{L/F}(\alpha)=(-1)^n a_0^e, \qquad \mathrm{Tr}_{L/F}(\alpha)=(-a_{d-1})^e.$$

Remark 2.5.6. $\mathrm{Tr}_{K/F}:K\to F$ is an F-linear map.

2.6. **Cyclic Extensions.**

Theorem 2.6.1 (Hilbert's Theorem 90 – multiplicative form)**.** *Let* K/F *be a cyclic extension of degree* n *and* $\mathrm{Gal}(K/F)=\langle\sigma\rangle$. *Then,* $\beta\in K$ *is such that* $\mathrm{N}_{K/F}(\beta)=1$ *if, and only if, there exists* $\alpha\in K^\times$ *such that* $\beta=\alpha/\sigma(\alpha)$.

Theorem 2.6.2. *Let* F *be a field such that* F *contains a* primitive n*-th root of unity for some fixed* n *not divisible by* $\mathrm{char}(F)$.

(1) *If* K/F *is cyclic of degree* n, *then* $K=F[\alpha]$ *where* α *is a root of* X^n-a, *for some* $a\in F$. *[In particular,* $\min_{\alpha,F}=X^n-a$.*]*

(2) *Conversely, if* $a\in F$ *and* α *is a root of* X^n-a, *then* $F[\alpha]/F$ *is cyclic, its degree, say* d, *is a divisor of* n, *and* $\alpha^d\in F$.

Remark 2.6.3. Note that, by linear independence of characters, if K/F is separable, then $\mathrm{Tr}_{K/F}$ is not constant equal to zero.

Theorem 2.6.4 (Hilbert's Theorem 90 – additive form)**.** *Let K/F be a cyclic extension of degree n and* $\mathrm{Gal}(K/F) = \langle \sigma \rangle$*. Then, $\beta \in K$ is such that* $\mathrm{Tr}_{K/F}(\beta) = 0$ *if, and only if, there exists $\alpha \in K^\times$ such that $\beta = \alpha - \sigma(\alpha)$.*

Theorem 2.6.5 (Artin-Schreier)**.** *Let F be a field of characteristic $p > 0$.*

1. *If K/F is cyclic of degree p, then $K = F[\alpha]$ where α is a root of $X^p - X - a$, for some $a \in F$. [In particular,* $\mathrm{min}_{\alpha,F} = X^p - X - a$*.]*
2. *Conversely, if $a \in F$ and $f \stackrel{\mathrm{def}}{=} X^p - X - a$, then either f splits completely in F or is irreducible over F. In the latter case, if α is a root of f, then $F[\alpha]/F$ is cyclic of degree p.*

2.7. **Solvable and Radical Extensions.**

Definition 2.7.1. A *finite* extension K/F is a *solvable extension* if it is separable and the normal closure L of K/F [which is then finite Galois over F] is such that $\mathrm{Gal}(L/F)$ is a solvable group.

Remark 2.7.2. Note that for a finite separable extension K/F to be solvable, it suffices that there exists some finite Galois extension of F containing K with its Galois group solvable.

Proposition 2.7.3. *The class of solvable extensions is distinguished.*

Definition 2.7.4.
1. A finite extension K/F is a *repeated radical extension* if there is a tower:
$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r = K,$$
such that $F_i = F_{i-1}[\alpha_i]$, where α_i is either a root of a polynomial $X^n - a$, for some $a \in F_{i-1}$ and with $\mathrm{char}(F) \nmid n$, or a root of $X^p - X - a$, for some $a \in F_{i-1}$, where $p = \mathrm{char}(F)$. [Note that α_i might then be a root of unity.]
2. A finite extension K/F is a *radical extension* if there is $L \supseteq K$ such that L/F is repeated radical.

Remark 2.7.5. Note that, by definition, if K is the splitting field of a separable polynomial $f \in F[X]$, then the roots of f are given by radicals [i.e., f is *solvable by radicals*] if, and only if, K is radical.

Proposition 2.7.6. *The class of radical extensions is distinguished.*

Theorem 2.7.7. *Let K/F be* separable. *Then, K/F is solvable if, and only if, it is radical.*

Remark 2.7.8. This allows us to determine when a polynomial can be solved by radicals simply by looking at its Galois group!

Theorem 2.7.9. *For $n = 2, 3, 4$ [and $\mathrm{char}(F) \neq 2, 3$] there are formulas for solving [general] polynomial equations of degree n by means of radicals. For $n \geq 5$, there aren't.*

Theorem 2.7.10. *Suppose that $f \in \mathbb{Q}[X]$ is irreducible and splits completely in $\mathbb{R}$. If any root of f lies in a* real *repeated radical extension of $\mathbb{Q}$, then $\deg f = 2^r$ for some non-negative integer r.*

Remark 2.7.11. Note that the above theorem tells us that we cannot replace *radical* by *repeated radical* in trying to express all roots of a polynomials in terms of radicals. For example, the polynomial $f = X^3 - 4X + 2$ splits completely in $\mathbb{R}$ and is solvable. So, we can write its roots in terms of radicals [since its radical], but we *must* have *complex numbers* to write them in terms of radicals [since is not repeated radical by the theorem above]. More precisely, if

$$\alpha \stackrel{\text{def}}{=} \sqrt[3]{\frac{\sqrt{111}}{9} - 1}, \qquad \text{and} \qquad \zeta_3 \stackrel{\text{def}}{=} \frac{\sqrt{3}}{2}\mathrm{i} - \frac{1}{2},$$

then the [all real] roots of f are

$$\alpha + \frac{4}{3\alpha}, \quad \alpha\,\zeta_3 + \frac{4}{3\alpha\,\zeta_3}, \quad \alpha\,\zeta_3^2 + \frac{4}{3\alpha\,\zeta_3^2}.$$

[We *cannot* rewrite the above roots only using radicals of real numbers!]

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