1) [10 points] Let:

$$\begin{aligned} R(x,y)&: \quad x \text{ is related to } y \\ S(x)&: \quad x \text{ is single} \\ H(x)&: \quad x \text{ is happy} \end{aligned}$$

Using the statements above, analyze the statement [which does not need to be true!]: "Every happy person is related to a person that is single".

Solution.

$$\forall x,\ H(x) \to (\exists y \text{ st } R(x,y) \wedge S(y))$$

□

2) [10 points] Rewrite the statement [about real numbers]:

$$\neg[\exists x \text{ st } \forall y,\ [(y > x) \to (\exists z \text{ st } z^2 = y)]]$$

as a positive statement [without the "¬"symbol].

Solution.

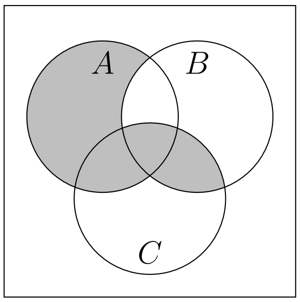
$$\begin{gathered} \neg[\exists x \text{ st } \forall y,\ [(y > x) \to (\exists z \text{ st } z^2 = y)]] \\ \sim \forall x,\ \neg[\forall y,\ [(y > x) \to (\exists z \text{ st } z^2 = y)]] \\ \sim \forall x,\ \exists y \text{ st } \neg[(y > x) \to (\exists z \text{ st } z^2 = y)]] \\ \sim \forall x,\ \exists y \text{ st } [(y > x) \wedge \neg(\exists z \text{ st } z^2 = y)] \\ \sim \forall x,\ \exists y \text{ st } [(y > x) \wedge (\forall z,\ z^2 \neq y)] \end{gathered}$$

□

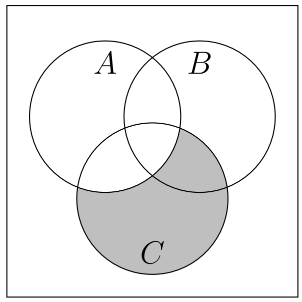
3) [20 points] Venn Diagrams:

(a) Shade the Venn Diagrams below.

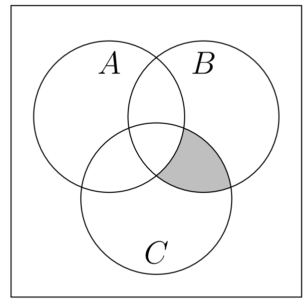
$(A \setminus B) \cup (B \cap C)$



$C \setminus A$

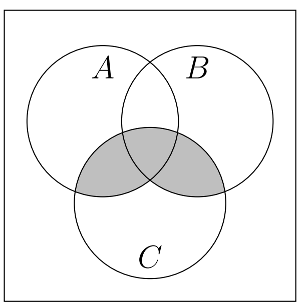


$[(A \setminus B) \cup (B \cap C)] \cap (C \setminus A)$



(b) Give concrete [and simple!] examples of sets A, B, C for which $[(A\setminus B)\cup(B\cap C)]\cap(C\setminus A)$ [the last set from (a)] is *not* the same as $(A \cap C) \cup (B \cap C)$. [**Hint:** Start by shading the Venn Diagram of the latter set.]

$(A \cap C) \cup (B \cap C)$



Let $A = B = C = \{1\}$. Then:

$$A \setminus B = \varnothing,$$
$$B \cap C = \{1\},$$
$$C \setminus A = \varnothing,$$
$$A \cap C = \{1\}.$$

So, $[(A \setminus B) \cup (B \cap C)] \cap (C \setminus A) = [\varnothing \cup \{1\}] \cap \varnothing = \varnothing$, while $(A \cap C) \cup (B \cap C) = \{1\} \cup \{1\} = \{1\}$, and hence they are different.

4) [20 points] Decide if the arguments below are valid by writing the [relevant parts of the] truth tables. [Remember that if an argument is *invalid*, then a single line of the table is enough!]

(a) If you are happy, your mother is happy. If your sister is happy, your mother is also happy. Your mother and your sister are not both happy. Therefore you are not happy.

Solution. Let:

P: you are happy
Q: your mother is happy
R: your sister is happy

So, the argument is:

$$\begin{array}{c} P \wedge Q \\ R \wedge Q \\ \neg(Q \wedge R) \\ \hline \therefore \neg P \end{array}$$

This argument is invalid, and hence it suffices to show one case when the premises are true but the conclusion is false. This is the case if P and Q are true but R is false.

$\square$

(b)

$$\begin{array}{c} P \wedge Q \to R \\ \neg P \to \neg Q \\ Q \\ \hline \therefore R \end{array}$$

Use the table below. [You don't need to fill it all, just what is necessary.]

Solution. We only need to fill a row until we find one "false" in the premises, as this would be an irrelevant row [as we only care if all premises are true]. The only case when this doesn't happen is when P, Q and R are all true, in which case the conclusion is also true. Therefore the argument is valid.

For sake of practice, I'll give the whole table below.

P	Q	R	$P \wedge Q \to R$	$\neg P \to \neg Q$	Q	R
T	T	T	T	T	T	$\boxed{T}$
T	T	F	$\boxed{F}$	T	T	F
T	F	T	T	T	$\boxed{F}$	T
T	F	F	T	T	$\boxed{F}$	F
F	T	T	T	$\boxed{F}$	T	T
F	T	F	T	$\boxed{F}$	T	F
F	F	T	T	T	$\boxed{F}$	T
F	F	F	T	T	$\boxed{F}$	F

Thus, the argument is <u>valid</u>.

$\square$

5) [20 points] Let A be a set and $\mathcal{F}$ be family of sets. Analyze $x \in (\bigcap \mathcal{F}) \setminus \mathscr{P}(A)$. [Your answer may only contain the symbols $\forall$, $\exists$, $\in$, $\notin$, $\vee$, $\wedge$, $\to$.]

Solution.

$$\begin{aligned} x \in (\bigcap \mathcal{F}) \setminus \mathscr{P}(A) &\sim [x \in (\bigcap \mathcal{F})] \wedge \neg[x \in \mathscr{P}(A)] \\ &\sim [\forall B \in \mathcal{F},\ x \in B] \wedge \neg[x \subseteq A] \\ &\sim [\forall B \in \mathcal{F},\ x \in B] \wedge \neg[\forall y \in x,\ y \in A] \\ &\sim [\forall B \in \mathcal{F},\ x \in B] \wedge [\exists y \in x \text{ st } y \notin A] \end{aligned}$$

If one does not want to use the shortcuts "$\forall B \in \mathcal{F}$" and "$\exists y \in x$", we get:

$$\begin{aligned} x \in (\bigcap \mathcal{F}) \setminus \mathscr{P}(A) &\sim [x \in (\bigcap \mathcal{F})] \wedge \neg[x \in \mathscr{P}(A)] \\ &\sim [\forall B,\ B \in \mathcal{F} \to x \in B] \wedge \neg[x \subseteq A] \\ &\sim [\forall B,\ B \in \mathcal{F} \to x \in B] \wedge \neg[\forall y,\ y \in x \to y \in A] \\ &\sim [\forall B,\ B \in \mathcal{F} \to x \in B] \wedge [\exists y \text{ st } \neg(y \in x \to y \in A)] \\ &\sim [\forall B,\ B \in \mathcal{F} \to x \in B] \wedge [\exists y \text{ st } y \in x \wedge y \notin A)] \end{aligned}$$

$\square$

6) [20 points] Let I be a non-empty set of indices and A_i and B_i be sets for each i in I. Show the equality below by analyzing what it means for an object x to be in each of the sets and using logical equivalences:

$$\bigcap_{i \in I} (A_i \setminus B_i) = \left(\bigcap_{i \in I} A_i \right) \setminus \left(\bigcup_{i \in I} B_i \right).$$

Solution. [This was a HW problem, with solution posted!] We have:

$$\begin{aligned}
x \in \bigcap_{i \in I} (A_i \setminus B_i) &\sim \forall i \in I,\ x \in (A_i \setminus B_i) \\
&\sim \forall i \in I,\ (x \in A_i) \land \neg (x \in B_i) \\
&\sim [\forall i \in I,\ x \in A_i] \land [\forall i \in I,\ \neg (x \in B_i)] \\
&\sim \left[x \in \bigcap_{i \in I} A_i \right] \land \neg \left[\exists i \in I \text{ st } x \in B_i \right] \\
&\sim \left[x \in \bigcap_{i \in I} A_i \right] \land \neg \left[x \in \bigcup_{i \in I} B_i \right] \\
&\sim x \in \left(\bigcap_{i \in I} A_i \right) \setminus \left(\bigcup_{i \in I} B_i \right).
\end{aligned}$$

For a solution without the shortcuts "$\forall i \in I$" and "$\exists i \in I$", check the posted HW solution.

$\square$