1) [8 points] Rewrite the statement [about real numbers]:

$$\neg[\forall x \in \mathbb{R},\ \exists y \in \mathbb{N} \text{ st } [(x \geq y) \to ((x + y > 0) \wedge (x = y + 2))]]$$

as a positive statement [without the "$\neg$"symbol].

Solution.

$$\exists x \in \mathbb{R} \text{ st } \forall y \in \mathbb{N},\ [(x \geq y) \wedge ((x + y \leq 0) \vee (x \neq y + 2))]$$

□

2) [8 points] Fill the truth table below.

P	Q	R	$P \wedge Q$	$(\neg Q) \vee R$	$(P \wedge Q) \to ((\neg Q) \vee R)$
T	T	T	T	T	T
F	T	T	F	T	T
T	T	F	T	F	F
F	T	F	F	F	T

3) [10 points] Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. Let $x \in A \cup (B \cap C)$. Then, $x \in A$ or $x \in B \cap C$.
If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, by definition of unions. Thus, $x \in (A \cup B) \cap (A \cup C)$, by definition of intersection.
If $x \in B \cap C$, then $x \in B$ and $x \in C$. Thus, the former tells us that $x \in A \cup B$, while the latter tells us that $x \in A \cup C$. Hence, $x \in (A \cup B) \cap (A \cup C)$.
Thus, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now, let $x \in (A \cup B) \cap (A \cup C)$. So, $x \in A \cup B$ and $x \in A \cup C$.
Suppose that $x \notin A$. Since $x \in A \cup B$, we have that either $x \in A$ or $x \in C$. Since $x \notin A$, we conclude that $x \in B$. Similarly, since $x \in A \cup C$, but $x \notin A$, we must have that $x \in C$. Therefore, $x \in B \cap C$.
Hence, either $x \in A$ or $x \in B \cap C$, i.e., $x \in A \cup (A \cap C)$. Thus, $(A \cup B) \cap (A \cup C) \subseteq A \cup (A \cap C)$.

Since we have both inclusions, we have $(A \cup B) \cap (A \cup C) = A \cup (A \cap C)$.

□

4) [10 points] Let $\mathcal{F}$ and $\mathcal{G}$ be a families of sets. Prove that $\bigcap(\mathcal{F} \cup \mathcal{G}) = (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Proof. Let $x \in \bigcap(\mathcal{F} \cup \mathcal{G})$. Thus, for all $A \in \mathcal{F} \cup \mathcal{G}$, we have that $x \in A$. In particular, if $A \in \mathcal{F}$, then $x \in A$ [as $\mathcal{F} \subseteq \mathcal{F} \cup \mathcal{G}$], and if $A \in \mathcal{G}$, then $x \in A$ [as $\mathcal{G} \subseteq \mathcal{F} \cup \mathcal{G}$]. The former means that $x \in \bigcap \mathcal{F}$, while the latter means that $x \in \cap \mathcal{G}$. Therefore, $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. Hence, $\bigcap(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Now, let $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. Then, $x \in \bigcap \mathcal{F}$ and $x \in \bigcap \mathcal{G}$. Now, let $A \in \mathcal{F} \cup \mathcal{G}$. Then, either $A \in \mathcal{F}$ or $A \in \mathcal{G}$. If the former holds, then $x \in A$, as $x \in \bigcap \mathcal{F}$ [by definition of the intersection of a family], and if the latter holds, then, similarly, we have that $x \in A$.
Thus, for all $A \in \mathcal{F} \cap \mathcal{G}$, we have that $x \in A$. Therefore, $x \in \bigcap(\mathcal{F} \cup \mathcal{G})$ [by definition]. Hence, $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) \subseteq \bigcap(\mathcal{F} \cup \mathcal{G})$.

Since we have both inclusions, the sets are equal.

□

5) [10 points] Let A be a set with partial order R and $a \in A$ the smallest element of A. Show that A has a unique minimal element. [What could this element be? In fact, we did this in class.]

Proof. This unique minimal element must be the smallest element. [I actually tell you that in the next problem!] So, that's what we will show.

[Remember, $x \in X$ is minimal if for all $y \in X$, yRx implies $y = x$. Also, $x \in X$ is the smallest element if for all $y \in X$, we have xRy.]

[a is minimal:] Let $b \in A$ and suppose that bRa. Since a is the least element, we have also that aRb [as $b \in A$]. Hence, since R is anti-symmetric, we have $a = b$, and hence a is minimal.

[a is the *unique* minimal:] Suppose $c \in A$ is minimal. Since a is the smallest element, we have that aRc. Thus, by definition of minimal, we have that $c = a$. Thus, every minimal element must be equal to a. $\square$

6) [12 points] Given $n \in \{1, 2, 3, 4, \ldots\}$, let $(0, 1/n)$ be [as usual in Calculus] the open interval of $\mathbb{R}$ given by $(0, 1/n) = \{x \in \mathbb{R} \,:\, 0 < x < 1/n\}$.
Let

$$\begin{aligned}\mathcal{F} &= \{\{0\}\} \cup \{(0, 1/n) \,:\, n \in \{1, 2, 3, 4, \ldots\}\} \\ &= \{\{0\}, (0, 1), (0, 1/2), (0, 1/3), (0, 1/4), \ldots\},\end{aligned}$$

and consider the partial order in $\mathcal{F}$ given by containment [as usual for sets].

(a) Show that $\{0\}$ is a minimal element of $\mathcal{F}$.

Proof. Suppose that $A \in \mathcal{F}$ is such that $A \subseteq \{0\}$. [We need to show $A = \{0\}$.] Then, since A has only one element, either $A = \varnothing$ or $A = \{0\}$. But the former cannot occur, as $\varnothing \notin \mathcal{F}$.

[Another way: if $A \in \mathcal{F}$, $A \subseteq \{0\}$, but $A \neq \{0\}$, then $A = (0, 1/n)$ for some n. But this is a contradiction, as $1/(2n) \in (0, 1/n)$, but $1/(2n) \notin \{0\}$.] □

(b) Show that for any $n \in \{1, 2, 3, \ldots\}$, $(0, 1/n)$ is *not* a minimal element of $\mathcal{F}$.

Proof. We have that $(0, 1/(n + 1)) \subseteq (0, 1/n)$, but $(0, 1/(n + 1)) \neq (0, 1/n)$. □

(c) Show that $\mathcal{F}$ has no smallest element. [**Hint:** Remember that if $A \in \mathcal{F}$ is a smallest element, then it is also a minimal element.]

Proof. Since the only minimal element is $\{0\}$ [as seen above], it would have to be the smallest element of $\mathcal{F}$ has such an element. But $\{0\} \not\subseteq (0, 1/2)$, so it is not the smallest element. Thus, $\mathcal{F}$ does not have a smallest element. □

[**Note:** This shows that a set can have only one minimal element, but no smallest element.]

7) [10 points] Let R be the equivalence relation on $\mathbb{R}$ given by aRb if $(a-b) \in \mathbb{Z}$. [You do *not* need to prove it is an equivalence relation.]

(a) Show that $[0]_R = \mathbb{Z}$. [Remember that $[0]_R$ is the equivalence class of 0 with respect to the relation R given above.]

Proof. We have:

$$\begin{aligned} x \in [0]_R \quad &\text{iff} \quad x \in \{y \in \mathbb{R} \,:\, yR0\} \\ &\text{iff} \quad x \in \{y \in \mathbb{R} \,:\, (y-0) \in \mathbb{Z}\} \\ &\text{iff} \quad x \in \{y \in \mathbb{R} \,:\, y \in \mathbb{Z}\} \\ &\text{iff} \quad x \in \mathbb{Z}. \end{aligned}$$

Thus, $[0]_R = \mathbb{Z}$. $\square$

(b) Find a real number x with $0 \leq x < 1$, such that $[2.31]_R = [x]_R$.

Solution. Remember: $[2.31]_R = [x]_R$ iff $xR2.31$.

We have that $x = 0.31$ is such that $0 \leq 0.31 < 1$ and $xR2.31$, as $x - 2.31 = -2 \in \mathbb{Z}$. $\square$

8) [12 points] Let R be an equivalence relation on a set A.

(a) Show that both $\text{Ran}(R)$ [the range of R] and $\text{Dom}(R)$ [the domain of R] are equal to A.

Proof. Let $a \in A$. Since $(a, a) \in R$ [as R is reflexive], we have that $a \in \text{Dom}(R)$ and $a \in \text{Ran}(R)$. So, A is contained in both. Since both domain and range are subsets of A by definition, we have the equalities. □

(b) Show that R^{-1} [the inverse relation] is equal to R.

Proof. Let $(a, b) \in R$. Since R is symmetric, we have that $(b, a) \in R$. Then, $(a, b) \in R^{-1}$ by definition of the inverse relation. Thus, $R \subseteq R^{-1}$.

Let $(a, b) \in R^{-1}$. Then, $(b, a) \in R$. Since R is symmetric, we have that $(a, b) \in R$. Thus, $R^{-1} \subseteq R$.

Since we have both inclusions, the sets must be equal. □

(c) Show that $R \circ R$ [the composition] is also equal to R.

Proof. Let $(a, c) \in R \circ R$. Then, by definition, there is $b \in A$ such that $(a, b), (b, c) \in R$. Since R is transitive, this means that $(a, c) \in R$. Hence, $R \circ R \subseteq R$.

Now, let $(a, b) \in R$. Since R is reflexive, we have that $(a, a) \in R$. Since then $(a, a), (a, b) \in R$, we have [by definition of composition] that $(a, b) \in R \circ R$. Thus, $R \subseteq R \circ R$.

Since we have both inclusions, the sets must be equal. □

9) [10 points] Prove that for $n \geq 0$ we have

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. We prove it by induction on n. For $n = 0$ we have that:

$$0 \cdot 1 = 0 = \frac{0 \cdot 1 \cdot 2}{3}.$$

Now assume that

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

Then,

$$\begin{aligned} 0 \cdot 1 + 1 \cdot 2 + \cdots + n \cdot (n+1) + (n+1) \cdot (n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= \left(\frac{n}{3} + 1\right)(n+1)(n+2) \\ &= \left(\frac{n+3}{3}\right)(n+1)(n+2) \\ &= \frac{(n+1)(n+2)(n+3)}{3}. \end{aligned}$$

Hence, the formula works for $(n+1)$, which finishes the induction. $\square$

10) [10 points] Remember that the Fibonacci sequence is given by:

$$F_0 = 0, \qquad F_1 = 1,$$

$$F_n = F_{n-2} + F_{n-1}, \text{ for } n \geq 2.$$

Consider now the recursively defined sequence given by

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1,$$

$$a_n = \frac{1}{2}a_{n-3} + \frac{3}{2}a_{n-2} + \frac{1}{2}a_{n-1}, \text{ for } n \geq 3.$$

Prove that $a_n = F_n$ for all $n \geq 0$.

[**Hint:** $\frac{3}{2}a_{n-2} = \frac{1}{2}a_{n-2} + a_{n-2}$.]

Proof. We prove it by [strong] induction on n. We need three first steps:

$$a_0 = 0 = F_0, \quad a_1 = 1 = F_1, \quad a_2 = 1 = 1 + 0 = F_2.$$

Assume now that for some $n \geq 2$ and all $k \leq n$ we have $a_k = F_k$. Then:

$$\begin{aligned}
a_{n+1} &= \frac{1}{2}a_{n-2} + \frac{3}{2}a_{n-1} + \frac{1}{2}a_n && \text{[recursive formula (as } n+1 \geq 3)] \\
&= \frac{1}{2}F_{n-2} + \frac{3}{2}F_{n-1} + \frac{1}{2}F_n && \text{[by ind. hyp.]} \\
&= \frac{1}{2}F_{n-2} + \frac{1}{2}F_{n-1} + F_{n-1} + \frac{1}{2}F_n && \text{[as in the hint]} \\
&= \frac{1}{2}(F_{n-2} + F_{n-1}) + F_{n-1} + \frac{1}{2}F_n && \text{[factor 1/2]} \\
&= \frac{1}{2}F_n + F_{n-1} + \frac{1}{2}F_n && \text{[recursive formula for } F_n] \\
&= F_n + F_{n-1} && \text{[add } (1/2)F_n\text{'s]} \\
&= F_{n+1} && \text{[recursive formula for } F_{n+1}]
\end{aligned}$$

Thus, the formula holds for $n + 1$, finishing the proof. □