Fixing Proof in Class

Math 552 – Spring 2015

April 22, 2015

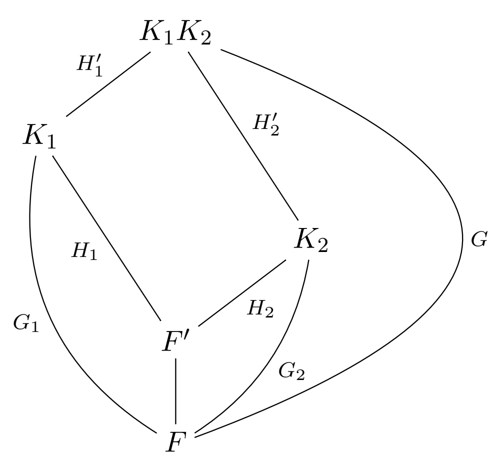
Theorem. *Suppose that K_i/F are Galois, with $G_i \stackrel{\text{def}}{=} \text{Gal}(K_i/F)$, for $i = 1, 2$. Let also, $G \stackrel{\text{def}}{=} \text{Gal}(K_1K_2/F)$, $F' \stackrel{\text{def}}{=} K_1 \cap K_2$ and $H_i \stackrel{\text{def}}{=} \text{Gal}(K_i/F')$. Then,*

$$\phi : G \to G_1 \times G_2,$$

defined by $\phi(\sigma) = (\sigma|_{K_1}, \sigma|_{K_2})$ is such that $\phi(G) \supseteq H_1 \times H_2$.

Proof. Let $\sigma_i \in H_i$ and $H_i' \stackrel{\text{def}}{=} \text{Gal}(K_1K_2/K_i) \leq G$. [We will show there is $\sigma \in G$ such that $\phi(\sigma) = (\sigma_1, \sigma_2)$, i.e., $\sigma|_{K_i} = \sigma_i$.]

We have:



By the *Natural Irrationalities Theorem*, we have that $H_i' \cong H_j$, where $i \neq j$, via the map $\tau \mapsto \tau|_{K_j}$.

So, since $\sigma_i \in H_i$, there is $\tilde{\sigma}_i \in H_j$ [for $j \neq i$], such that $\tilde{\sigma}_i|_{K_i} = \sigma_i$. Let $\sigma \stackrel{\text{def}}{=} \tilde{\sigma}_1 \circ \tilde{\sigma}_2$.

Let $\alpha_1 \in K_1$. Then, since $\tilde{\sigma}_2 \in H_1' = \text{Gal}(K_1K_2/K_1)$, we have that $\tilde{\sigma}_2(\alpha_1) = \alpha_1$. Thus, $\sigma(\alpha_1) = \tilde{\sigma}_1(\tilde{\sigma}_2(\alpha_1)) = \tilde{\sigma}_1(\alpha_1) = \sigma_1(\alpha_1)$. Thus, $\sigma|_{K_1} = \sigma_1$.

If now $\alpha_2 \in K_2$, then $\tilde{\sigma}_2(\alpha_2) = \sigma_2(\alpha_2) \in K_2$, as σ_2 is an automorphism of K_2. But since $\tilde{\sigma}_1$ fixes K_2 [as it is in $H_2' = \text{Gal}(K_1K_2/K_2)$], we have that $\sigma(\alpha_2) = \tilde{\sigma}_1(\tilde{\sigma}_2(\alpha_2)) = \tilde{\sigma}_1(\sigma_2(\alpha_2)) = \sigma_2(\alpha_2)$. Thus, $\sigma|_{K_2} = \sigma_2$. ∎