1. [August 1996 Prelim, Part 4, Question 1] Prove that $\mathrm{Aut}(\mathbb{C})$ is uncountable.

   Proof. To do this, we need to "tweak" are result from class.

   Lemma. *Let K/F be a field extension, with K algebraically closed, and $\sigma \in \mathrm{Aut}(F)$. Then, there exists $\tilde{\sigma} \in \mathrm{Aut}(K)$ such that $\tilde{\sigma}|_F = \sigma$ [i.e., we can extend σ to K].*

   Proof. We've basically done it for *algebraic* extensions. Let $\mathcal{S}$ be the set of all pairs (E, τ) where E is an intermediate extension of K/F and $\tau \in \mathrm{Aut}(E)$ that extends σ. We use Zorn's Lemma to get a maximal element $(E, \tau) \in \mathcal{S}$. [This is just like what we did in class!]

   We now prove that $E = K$. If not, let $\alpha \in K \setminus E$. If α is *transcendental* [i.e., not algebraic] over F, then $\tilde{\tau} : E[\alpha] \to E[\alpha]$ such that $\tilde{\tau}(\alpha) = \alpha$ and $\tilde{\tau}|_E = \tau$ is an automorphism of $E[\alpha]$, that extends τ, contradicting the maximality of (E, τ). Thus, we may assume K/E is algebraic.

   Now, if K/E is algebraic, and since K is algebraically closed, then K is the algebraic closure of E. We can then extend τ to an *embedding* $\tilde{\tau} : K \to K$. But, K being algebraically closed, we get that $\tilde{\tau}$ is also onto [also done in class], so $\tilde{\tau} \in \mathrm{Aut}(K)$. This, again, contradicts the maximality of (E, τ). □

   Now, we know that $T = \{t \in \mathbb{C} \ : \ t \text{ is transcendental over } \mathbb{Q}\}$ is uncountable. Take $t_1, t_2 \in T$ with $t_1 \neq t_2$, and let $F \stackrel{\text{def}}{=} \mathbb{Q}(t_1, t_2)$. Then, we have that $\sigma : F \to F$, given by $\sigma(t_1) = t_2$, $\sigma(t_2) = t_1$ and $\sigma|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}$ is an automorphism of F. By the lemma, it can be extended to $\mathbb{C}$. Since we have uncountably many different σ's constructed this way [using the uncountably many elements of T], we get uncountably many elements in $\mathrm{Aut}(\mathbb{C})$. □

2. [August 2006 Prelim, Part IV, Question 1] Let K/F be an infinite extension. Show that there exists an infinite chain of intermediate fields between K and F.

   Proof. Suppose that K/F is algebraic. Let $\alpha_1 \in K \setminus F$. Definite $F_1 \stackrel{\text{def}}{=} F[\alpha_1]$. [Since $\alpha_1 \notin F$, $F_1 \neq F$.]

   Since F_1/F is finite, we have that K/F_1 is infinite and we can take $\alpha_2 \in F \setminus F_1$. Then, let $F_2 \stackrel{\text{def}}{=} F_1[\alpha_2]$, and proceed inductively. We then obtain a chain:

$$F \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subseteq K.$$

If K/F is not algebraic, we gave an transcendental element $t \in K$. Then, we get that

$$F \subseteq \cdots F(t^8) \subsetneq F(t^4) \subsetneq F(t^2) \subsetneq F(t) \subseteq K.$$

$\square$

3. Example of *finte* extension K/F with infinitely many intermediate extensions.

   Proof. Let $K = \mathbb{F}_p(s,t)$ and $F = \mathbb{F}_p(s^p, t^p)$. Then, we have that $[K : F] = p^2$. [Check!] Also, K/F has no primitive element. If $\alpha \in K$, then α is a rational function on s and t and hence a^p is a rational function on s^p and t^p, so is in F. Thus, for all $\alpha \in K$, $[F[\alpha] : F] \leq p < p^2 = [K : F]$, so $F[\alpha] \neq K$.

   Indeed, we have infinitely many intermediate extensions: let $E_i \stackrel{\text{def}}{=} F[s + s^{ip}t]$, for $i \in \{1, 2, 3, \ldots\}$. If $E_i = E_j$, for $i \neq j$, say both equal E, then $(s + s^{ip}t) - (s + s^{jp}t) = t(s^{ip} - s^{jp}) \in E$. Since $F \subseteq E$, we get that also $(s^{ip} - s^{jp}) \in E$, and thus the quotient of these elements, namely t, is also in E.

   So, we have that $t, s^{ip}, s + s^{ip}t \in E$, and thus $s \in E$. But then $s, t \in E$ and therefore $E = K$. But this is a contradiction, as E/F has a primitive element [e.g., $s + s^{ip}t$].

   So, the extensions E_i are all distinct, giving infinitely many intermediate extensions between F and K. $\square$