Errata 3

Math 652

April 20, 2016

In class I've said that if (K, u) is a complete valued field with u not discrete, then for $a_n \in \mathbb{Q}_{>0}$ we have that $u(a_n) = a_n$. Although this is true for completions of number fields with their normalized absolute value [in which case all of this would be trivial], it is not true in general.

What we have is the following: given n [and x_n] we have that there is $a_n \in K^\times$ such that

$$0 < |a_n| - \frac{1}{\|x_n\|} < \frac{\epsilon}{\|x_n\|},$$

since $u = |\ \cdot\ |$ is not discrete [and hence, by the Lemma below, $|K^\times|$ is dense in $\mathbb{R}_{>0}$]. So, replacing a_n by $|a_n|$ in what we've done in class we have that

$$1 < |a_n| \|x_n\| \leq 1 + \epsilon,$$

i.e.,

$$1 < \| \underbrace{a_n x_n}_{\tilde{x}_n} \| \leq 1 + \epsilon.$$

The rest remains the same.

Here is the necessary lemma:

Lemma. *If $(K, |\cdot|)$ is a valued field [not necessarily complete] and $|\cdot|$ is not discrete, then $|K^\times|$ is dense in $\mathbb{R}_{>0}$.*

Proof. Since $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is a isomorphism of groups [$\mathbb{R}_{>0}$ with multiplication and $\mathbb{R}$ with addition] and also *continuous* [actually a homeomorphism with inverse e^x], it suffices to show that the additive group $G \stackrel{\text{def}}{=} \log(|K^\times|)$ is dense in $\mathbb{R}$. [This makes things a little easier, since it's easier to deal with convergence in an additive setting.]

We will show, in general, that if G [an subgroup of $\mathbb{R}$] has an accumulation point, then it is dense in $\mathbb{R}$.

By hypothesis, G has an accumulation point, say a. Hence there is either a strictly increasing sequence or a strictly decreasing sequence $\{a_n\}$, with $a_n \in G$ such that $a_n \to a$. In either case, we have that the sequences $\{a_{n+1} - a_n\}$ and $\{a_n - a_{n+1}\}$ are sequences of elements of G that converge to 0, one from the left and the other from the right. So 0 is an accumulation point, with sequences in G that converge to it from both sides. Let then $\{r_n\}$ then be a strictly increasing sequence in G converging to 0 and $\{s_n\}$ a strictly decreasing sequence in G converging to 0.

Now let $a \in \mathbb{R}$ [an arbitrary point]. We need to show that a is an accumulation point of G. If $a \in G$, then $\{a + r_n\}$ is a strictly increasing sequence of elements of G converging to a and so a is an accumulation point.

Suppose then that $a \notin G$ and let $b = \sup\{x \in G \ : \ x < a\}$. [Note that this b is well defined, since $|\cdot|$ is non-trivial and hence there are arbitrarily small [and arbitrarily large] elements in G.]

If $b = a$, then since $b = a \notin G$, we have that a is an accumulation point [by the definition of b].

So, suppose that $b \neq a$ and let $\epsilon \stackrel{\text{def}}{=} a - b > 0$. Since $s_n \to 0$ from the right, there is N such that $0 < s_N < \epsilon$. Also, by definition of b, there is $b_0 \in G$ such that $0 \leq b - b_0 < s_N/2$. [Note that we then have $b_0 \leq b < a$.] Hence, we have:

$$(s_N + b_0) - b = s_N - (b - b_0) > s_N - \frac{s_N}{2} = \frac{s_N}{2} > 0.$$

Thus, $s_N + b_0 > b$.

Also,

$$s_N < \epsilon = a - b \leq a - b_0,$$

and thus, we have that $s_N + a_0 < a$.

So, we have $s_N + b_0 \in G$ [since $s_N, b_0 \in G$] and $b < s_N + b_0 < a$. But this contradicts the definition of b, so this case [$a \notin G$ and $a \neq b$] cannot happen. Therefore, a is an accumulation point [since it is in every other case].

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